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6.055J / 2.038J The Art of Approximation in Science and Engineering
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An error of 15 percent is often small compared to the other inaccuracies in an approximate computation, so this method of approximate minimization is a valuable time-saver.

Now return to the original problem: determining the Bohr radius. The approximate minimization predicts the correct value. Even if the method were not so charmed, there is no point in doing a proper, calculus minimization. The calculus method is *too accurate* given the inaccuracies in the rest of the derivation.

Engineers understand this idea of not over-engineering a system. If a bicycle most often breaks at welds in the frame, there is little point replacing the metal between the welds with expensive, high-strength aerospace materials. The new materials might last 100 years instead of 50 years, but such a replacement would be overengineering. To improve a bicycle, put effort into improving or doing without the welds.

In estimating the Bohr radius, the kinetic-energy estimate uses a crude form of the uncertainty principle, $\Delta p \Delta x \sim \hbar$, whereas the true statement is that $\Delta p \Delta x \geq \hbar/2$. The estimate also uses the approximation $E_{\text{Kinetic}} \sim (\Delta p)^2/m$. This approximation contains m instead of $2m$ in the denominator. It also assumes that Δp can be converted into an energy as though it were a true momentum rather than merely a crude estimate for the root-mean-square momentum. The potential- and kinetic-energy estimates use a crude definition of position uncertainty Δx : that $\Delta x \sim r$. After making so many approximations, it is pointless to minimize the result using the elephant gun of differential calculus. The approximate method is as accurate as, or perhaps more accurate than the approximations in the energy.

This method of equating competing terms is **balancing**. We balanced the kinetic energy against the potential energy by assuming that they are roughly the same size. The consequence is that

$$a_0 \sim \frac{\hbar^2}{m_e(e^2/4\pi\epsilon_0)}.$$

Nature could have been unkind: The potential and kinetic energies could have differed by a factor of 10 or 100. But Nature is kind: The two energies are roughly equal, except for a constant that is nearly 1. ‘Nearly 1’ is also called **of order unity**. This rough equality occurs in many examples, and you often get a reasonable answer by pretending that two energies (or two quantities with the same units) are equal. When the quantities are potential and kinetic energy, as they often are, you get extra safety: The so-called virial theorem protects you against large errors (for more on the virial theorem, see any intermediate textbook on classical dynamics).

7.5 Bending of light by gravity

Rocks, birds, and people feel the effect of gravity. So why not light? The analysis of that question is a triumph of Einstein’s theory of general relativity. I can calculate how gravity bends light by solving the so-called geodesic equations from general relativity:

$$\frac{d^2 x^\beta}{d\lambda^2} + \Gamma^{\beta}_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0.$$

To compute the Christoffel symbols $\Gamma_{\mu\nu}^{\beta}$ requires solving for the metric tensor $g_{\mu\nu}$, which requires solving the curvature equations $R_{\mu\nu} = 0$.

The curvature equations are a shorthand for ten partial-differential equations. The equations are rich in mathematical interest but are a nightmare to solve. The equations are numerous – that’s one problem – but worse, they are not linear. So the standard trick, which is to guess a type of solution and form new solutions by combining the basic types, does not work. You can spend a decade learning advanced mathematics to solve the equations exactly. Or you can accept the great principle of analysis: When the going gets tough, lower your standards. If I sacrifice accuracy, I can explain light bending in less than one thousand pages using mathematics and physics that you (and I!) already know.

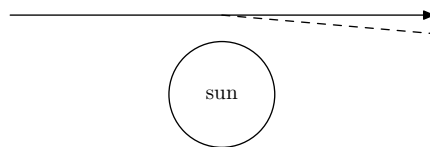
The simpler method is dimensional analysis, in the usual three steps:

1. Find the relevant parameters.
2. Find dimensionless groups.
3. Use the groups to make the most general dimensionless statement.
4. Add physical knowledge to narrow the possibilities.

The following sections do each step.

7.5.1 Finding parameters

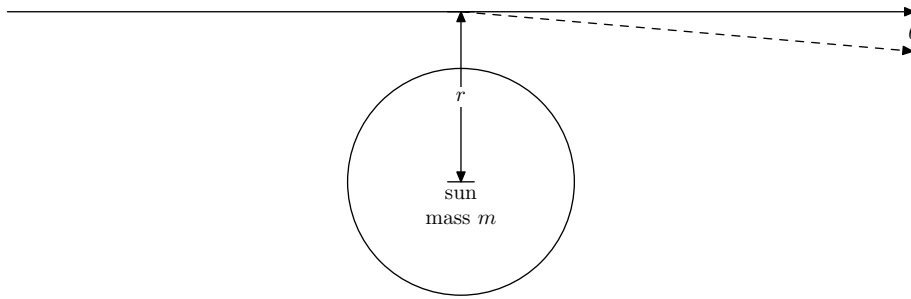
The first step in a dimensional analysis is to decide what physical parameters the bending angle can depend on. An unlabeled diagram prods me into thinking of labels, many of which are parameters of the problem:



Here are reasons to include various parameters:

1. The list has to include the quantity to solve for. So the angle θ is the first item in the list.
2. The mass of the sun, m , has to affect the angle. Black holes greatly deflect light, probably because of their huge mass.
3. A faraway sun or black hole cannot strongly affect the path (near the earth light seems to travel straight, in spite of black holes all over the universe); therefore r , the distance from the center of the mass, is a relevant parameter. The phrase ‘distance from the center’ is ambiguous, since the light is at various distances from the center. Let r be the distance of closest approach.
4. The dimensional analysis needs to know that gravity produces the bending. The parameters listed so far do not create any forces. So include Newton’s gravitational constant G .

Here is the same diagram with important parameters labeled:



Here is a table of the parameters and their dimensions:

<i>Parameter</i>	<i>Meaning</i>	<i>Dimensions</i>
θ	angle	–
m	mass of sun	M
G	Newton's constant	$L^3T^{-2}M^{-1}$
r	distance from center of sun	L

where, as you might suspect, L, M, and T represent the dimensions of length, mass, and time, respectively.

7.5.2 Dimensionless groups

What are the dimensionless groups? The parameter θ is an angle, which is already dimensionless. The other variables, G , m , and r , cannot form a second dimensionless group. To see why, following the dimensions of mass. It appears only in G and m , so a dimensionless group would contain the product Gm , which has no mass dimensions in it. But Gm and r cannot get rid of the time dimensions. So there is only one independent dimensionless group, for which θ is the simplest choice.

I want a second dimensionless group because otherwise my analysis seems like nonsense. Any physical solution can be written in dimensionless form; this idea is the foundation of dimensional analysis. With only one dimensionless group, θ , I have to conclude that θ depends on no variables at all:

$$\theta = \text{function of other dimensionless groups,}$$

but there are no other dimensionless groups, so

$$\theta = \text{constant.}$$

This conclusion is crazy! The angle must depend on at least one of m and r . My physical picture might be confused, but it's not so confused that neither variable is relevant. So I need to make another dimensionless group on which θ can depend. Therefore, I return to Step 1: Finding parameters.

The list so far lacks a crucial parameter.

What physics have I neglected? Free associating often suggests the missing parameter. Unlike rocks, light is difficult to deflect, otherwise humanity would not have waited until

the 1800s to study the deflection, whereas the path of rocks was studied at least as far back as Aristotle and probably for millions of years beforehand. Light travels much faster than rocks, which may explain why light is so difficult to deflect: The gravitational field 'gets hold of it' only for a short time. But none of my parameters distinguish between light and rocks. Therefore I should include c , the speed of light. It introduces the fact that I'm studying light, and it does so with a useful distinguishing parameter, the speed.

Here is the latest table of parameters and dimensions:

<i>Parameter</i>	<i>Meaning</i>	<i>Dimensions</i>
θ	angle	–
m	mass of sun	M
G	Newton's constant	$L^3T^{-2}M^{-1}$
r	distance from center of sun	L
c	speed of light	LT^{-1}

Length is strewn all over the parameters (it's in G , r , and c). Mass, however, appears in only G and m , so I already know I need a combination such as Gm to cancel out mass. Time also appears in only two parameters: G and c . To cancel out time, I need to form Gm/c^2 . This combination has one length in it, so a dimensionless group is Gm/rc^2 .

7.5.3 Drawing conclusions

The most general relation between the two dimensionless groups is

$$\theta = f\left(\frac{Gm}{rc^2}\right).$$

Dimensional analysis cannot tell me the correct function f .

Physical reasoning and symmetry narrow the possibilities. First, strong gravity – from a large G or m – should increase the angle. So f should be an increasing function. Now try symmetry: Imagine a world where gravity is repulsive or, equivalently, the gravitational constant is negative. Then the angle should also be negative, so f should be an odd function. This symmetry argument eliminates choices like $f(Gm/rc^2) \sim (Gm/rc^2)^2$.

The simplest guess is that f is the identity function. Then the bending angle is

$$\theta = \frac{Gm}{rc^2}.$$

There is likely a dimensionless constant in f :

$$\theta = 7 \frac{Gm}{rc^2}$$

or

$$\theta = 0.3 \frac{Gm}{rc^2}$$

are also possible. This freedom means

$$\theta \sim \frac{Gm}{rc^2}.$$

7.5.4 Comparison with exact calculations

Different theories of gravity give the same result

$$\theta \sim \frac{Gm}{rc^2};$$

the only variation is in the value for the missing dimensionless constant. Here are those values from exact calculation:

$$\theta = \frac{Gm}{rc^2} \times \begin{cases} 1 & \text{(simplest guess);} \\ 2 & \text{(Newtonian gravity);} \\ 4 & \text{(Einstein's theory).} \end{cases}$$

Here is a rough explanation of the origin of those constants. The 1 for the simplest guess is just that. The 2 for Newtonian gravity is from integrating angular factors like cosine and sine that determine the position of the photon as it moves toward and past the sun.

The most interesting constant is the 4 for general relativity, which is twice the Newtonian value because light moves at the speed of light. The extra bending is a consequence of Einstein's theory of special relativity putting space and time on the same level. The theory of general relativity then formulates gravity in terms of the curvature of spacetime. Newton's theory is the limit of general relativity that considers only time curvature; general relativity itself also calculates the space curvature. Since most objects move much slower than the speed of light, meaning that they travel much farther in time than in space, they feel mostly the time curvature. The Newtonian analysis is fine for those objects. Since light moves at the speed of light, it sees equal amounts of space and time curvature, so it bends twice as far as the Newtonian theory would predict.

7.5.5 Numbers!

At the surface of the Earth, the strength is

$$\frac{Gm}{rc^2} \sim \frac{6.7 \cdot 10^{-11} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1} \times 6.0 \cdot 10^{24} \text{ kg}}{6.4 \cdot 10^6 \text{ m} \times 3.0 \cdot 10^8 \text{ m s}^{-1} \times 3.0 \cdot 10^8 \text{ m s}^{-1}} \sim 10^{-9}.$$

This miniscule value is the bending angle (in radians). So if physicists want to show that light bends, they had better look beyond the earth! That statement is based on another piece of dimensional analysis and physical reasoning, whose result I quote without proof: A telescope with mirror of diameter d can resolve angles roughly as small as λ/d , where λ is the wavelength of light. One way to measure the bending of light is to measure the change in position of the stars. A lens that could resolve an angle of 10^{-9} has a diameter of at least

$$d \sim \lambda/\theta \sim \frac{0.5 \cdot 10^{-6} \text{ m}}{10^{-9}} \sim 500 \text{ m}.$$

Large lenses warp and crack; one of the largest lenses made is 6 m. So there is no chance of detecting an angle of 10^{-9} .

Physicists therefore searched for another source of light bending. In the solar system, the largest mass is the sun. At the surface of the sun, the field strength is

$$\frac{Gm}{rc^2} \sim \frac{6.7 \cdot 10^{-11} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1} \times 2.0 \cdot 10^{30} \text{ kg}}{7.0 \cdot 10^8 \text{ m} \times 3.0 \cdot 10^8 \text{ m s}^{-1} \times 3.0 \cdot 10^8 \text{ m s}^{-1}} \sim 2.1 \cdot 10^{-6} \approx 0.4''.$$

This angle, though small, is possible to detect: The required lens diameter is roughly

$$d \sim \lambda/\theta \sim \frac{0.5 \cdot 10^{-6} \text{ m}}{2.1 \cdot 10^{-6}} \sim 20 \text{ cm}.$$

The eclipse expedition of 1919, led by Arthur Eddington of Cambridge, tried to measure exactly this effect.

For many years Einstein believed that his theory of gravity would predict the Newtonian value, which turns out to be 0.87 arcseconds for light just grazing the surface of the sun. The German mathematician, Soldner, derived the same result in 1803. Fortunately for Einstein's reputation, the eclipse expeditions that went to test his (and Soldner's) prediction got rained or clouded out. By the time an expedition got lucky with the weather (Eddington's in 1919), Einstein had invented a new theory of gravity, which predicted 1.75 arcseconds. The goal of Eddington's expedition was to decide between the Newtonian and general relativity values. The measurements are difficult, and the results were not accurate enough to decide which theory was right. But 1919 was the first year after the World War, in which Germany and Britain had fought each other almost to oblivion. A theory invented by a German, confirmed by an Englishman (from Newton's university, no less) – such a picture was comforting after the trauma of war, so the world press and scientific community saw what they wanted to: Einstein vindicated! A proper confirmation of Einstein's prediction came only with the advent of radio astronomy, which could measure small deflections accurately. I leave you with this puzzle: If the accuracy of a telescope is λ/d , how could radio telescopes be more accurate than optical ones, since radio waves have a longer wavelength than light has?!

7.6 Buckingham Pi theorem

The second step in a dimensional analysis is to make dimensionless groups. That task is simpler by knowing in advance how many groups to look for. The Buckingham Pi theorem provides that number. I derive it with a series of examples.

Here is a possible beginning of the theorem statement: *The number of dimensionless groups is...* Try it on the light-bending example. How many groups can the variables θ , G , m , r , and c produce? The possibilities include θ , θ^2 , Gm/rc^2 , $\theta Gm/rc^2$, and so on. The possibilities are infinite! Now apply the theorem statement to estimating the size of hydrogen, before including quantum mechanics in the list of variables. That list is a_0 (the size), $e^2/4\pi\epsilon_0$, and m_e . That list produces no dimensionless groups. So it seems that the number of groups would be zero – if no groups are possible – or infinity, if even one group is possible.