LECTURE 10

- Readings: Section 3.6

Lecture outline

- More on continuous r.v.s
- Derived distributions

Review

Discrete	Continuous
$p_X(x)$	$f_X(x)$
$p_{X,Y}(x,y)$	$f_{X,Y}(x,y)$
$p_{X\|Y}(x\|y) = \dfrac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\|Y}(x\|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$
$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$

$$F_X(x) = \mathbf{P}(X \leq x)$$

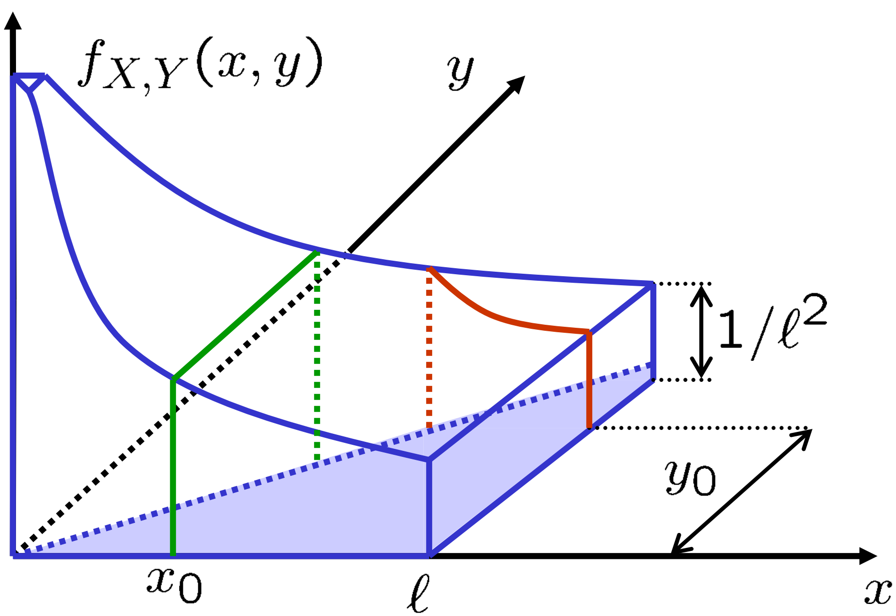
$$\mathbf{E}[X], \text{ var}(X)$$

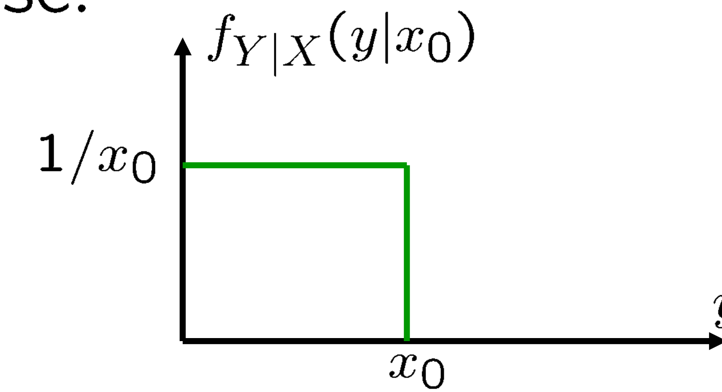
Conditioning "slices" the joint PDF

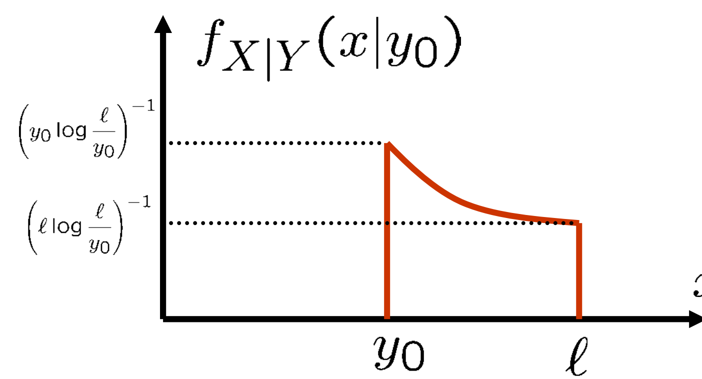
- Recall the stick-breaking example:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\ell x} & 0 \le y < x \le \ell \\ 0 & \text{otherwise.} \end{cases}$$

- Pictorially:

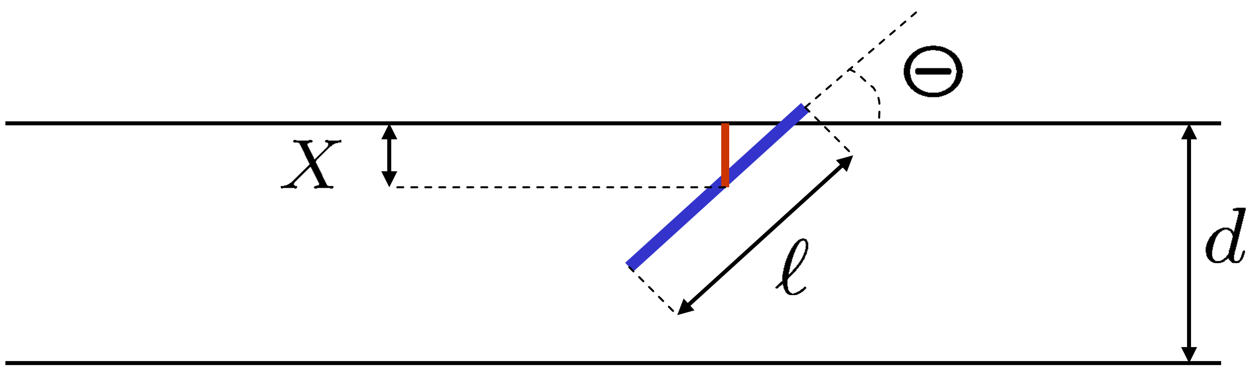






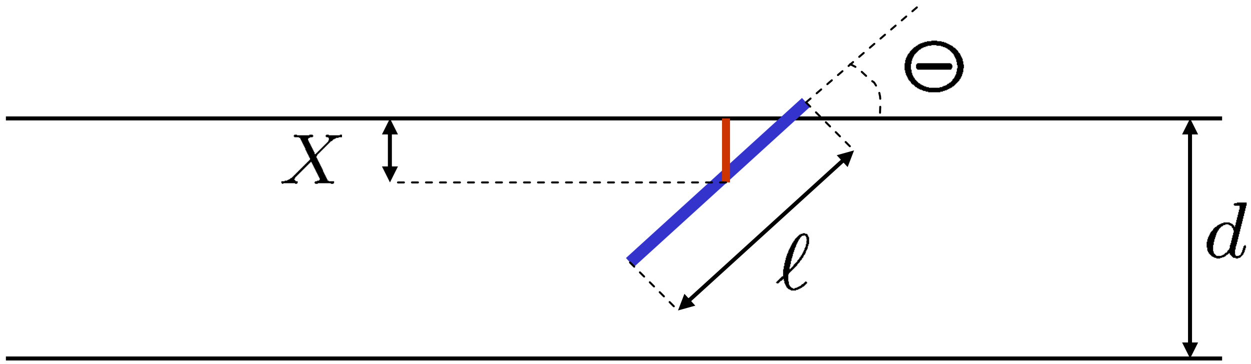
Buffon's Needle (1)

- Parallel lines at distance d
  Needle of length ℓ (assume $\ell < d$)
- Find $\mathbf{P}$(needle intersects one of the lines).



- Midpoint-nearest line distance: $X \in [0, d/2]$
- Needle-lines acute angle: $\Theta \in [0, \pi/2]$

Buffon's Needle (2)



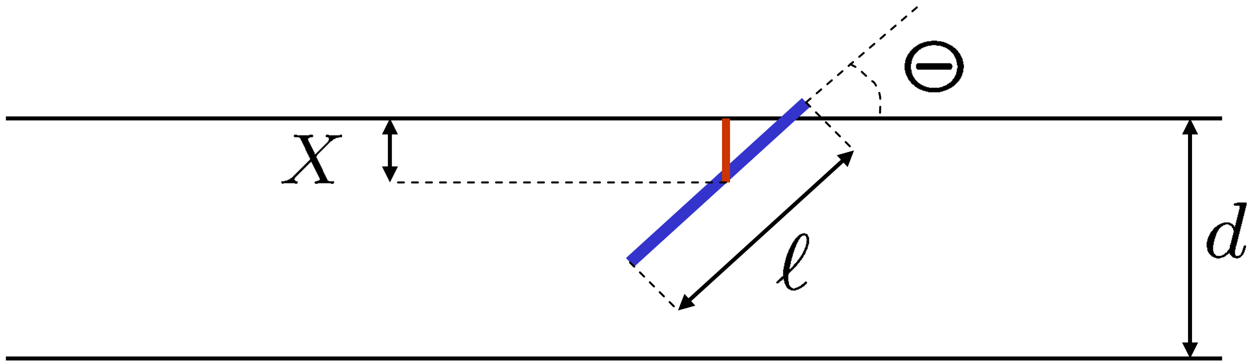
- Model: X, Θ uniform and independent.

$$f_{X,\Theta}(x,\theta) = f_X(x) \cdot f_\Theta(\theta)$$
$$= \frac{2}{d} \cdot \frac{4}{\pi} \qquad 0 \le x \le d/2,\ 0 \le \theta \le \pi/2$$

- When does the needle intersect a line?

If $X \le \frac{\ell}{2} \sin \Theta$

Buffon's Needle (3)



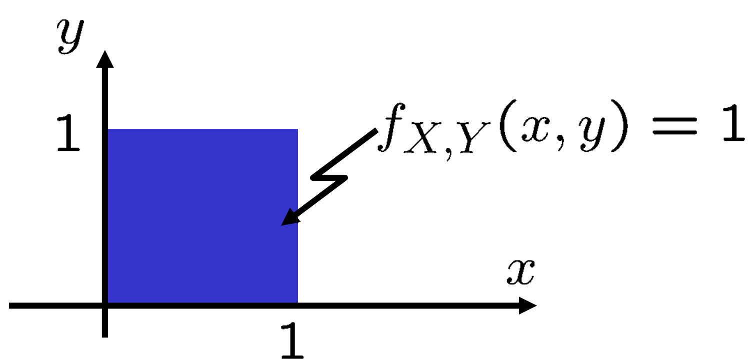
$$\mathbf{P}\left(X \leq \frac{\ell}{2}\sin\Theta\right) = \int\int_{x\leq\frac{\ell}{2}\sin\theta} f_X(x) f_\Theta(\theta)\,dx\,d\theta$$

$$= \frac{4}{\pi d}\int_0^{\pi/2}\int_0^{(\ell/2)\sin\theta} dx\,d\theta$$

$$= \frac{4}{\pi d}\int_0^{\pi/2}\frac{\ell}{2}\sin\theta\,d\theta = \frac{2\ell}{\pi d}$$

What is a derived distribution?

- It is a PMF or PDF of a function of random variables with known probability law.
- Example: X and Y



- Let: $g(X,Y) = Y/X$. Note: $g(X,Y)$ is a r.v.
- Obtaining the PDF for $g(X,Y)$ involves deriving a distribution.

Why do we derive distributions?

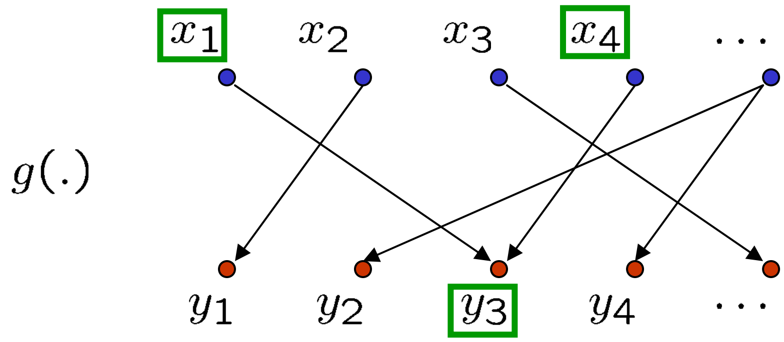
- Sometimes we don't need to. Example:
  - Computing expected values.

$$\mathbf{E}[g(X,Y)] = \iint g(x,y) f_{X,Y}(x,y) dx dy$$

- But often they're useful. Examples:
  - Maximum of several r.v.s. (delay models)
  - Minimum of several r.v.s (failure models).
  - Sum of several r.v.s. (multiple arrivals)

How to find them: Discrete Case

- Consider: - a single discrete r.v.: X
  - and a function: $g(X) = Y$



- Obtain probability mass for each possible value of $Y = y$:

$$p_Y(y) = \mathrm{P}(g(X) = y)$$
$$= \sum_{x:\, g(x)=y} p_X(x)$$

How to find them: Continuous Case

- Consider: - a single continuous r.v.: X
  - and a function: $g(X) = Y$

- Two step procedure:

  1. Get CDF of Y : $F_Y(y) = \mathbf{P}(Y \leq y)$

  2. Differentiate to get: $f_Y(y) = \frac{dF_Y}{dy}(y)$

- Why go to the CDF?

Example 1

- X: uniform on $[0,2]$
- Find PDF of $Y = X^3$
- **Solution**:

1. Get the CDF:

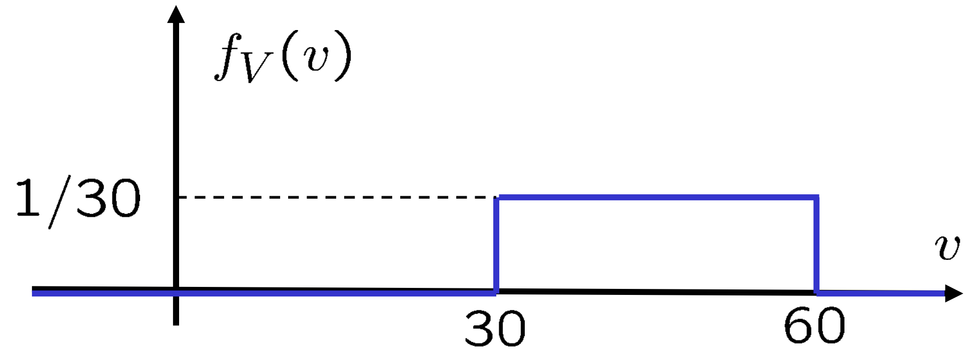
$$F_Y(y) = \mathbf{P}(Y \leq y) = \mathbf{P}(X^3 \leq y)$$
$$= \mathbf{P}(X \leq y^{1/3}) = \frac{1}{2}y^{1/3}$$

2. Differentiate:

$$f_Y(y) = \frac{dF_Y}{dy}(y) = \frac{1}{6y^{2/3}}$$

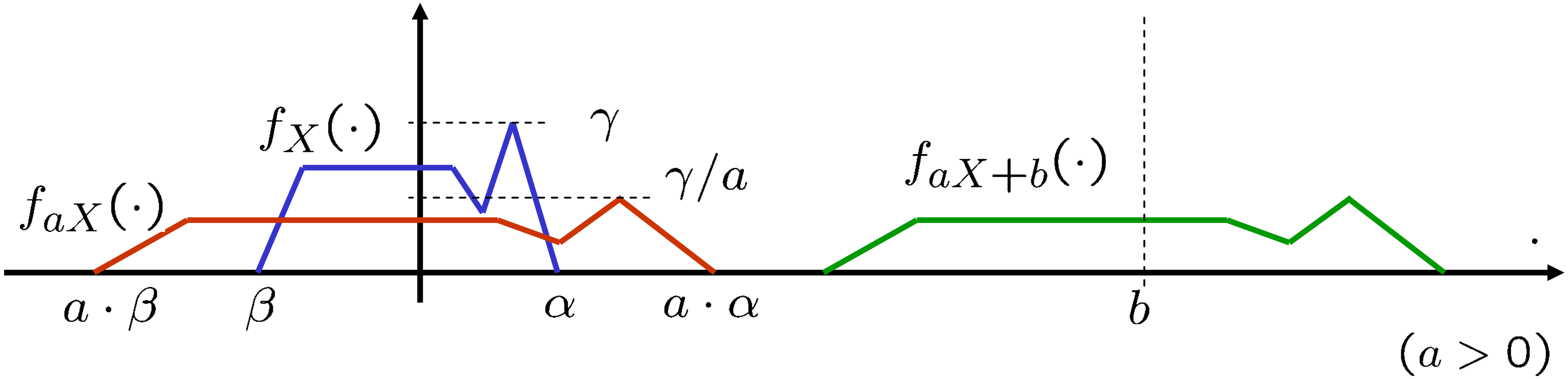
Example 2

- Joan is driving from Boston to New York. Her speed is uniformly distributed between 30 and 60 mph. What is the distribution of the duration of the trip?
- PDF of the velocity V:



- Let: $T(V) = \frac{200}{V}$
- Find $f_T(t)$.

The PDF of $Y = aX + b$.



$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

- Use this to check that if X is normal, then $Y = aX + b$ is also normal.