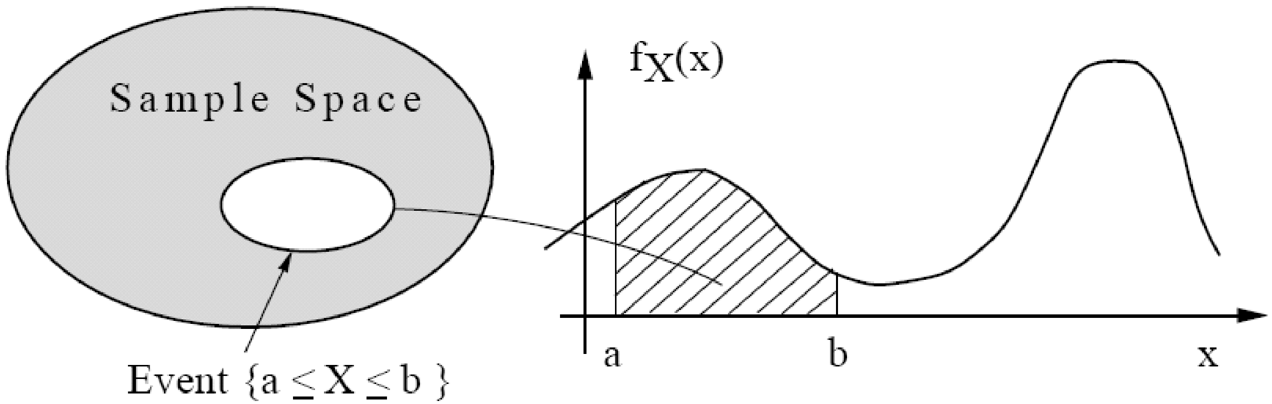
LECTURE 8

- Readings: Section 3.1-3.3

Lecture outline

- Probability density functions
- Cumulative distribution functions
- Normal random variables

Continuous Random Variables
Probability Density Function (PDF)



$$\mathbf{P}(a \leq X \leq b) = \int_a^b f_X(x)dx$$

- $\mathbf{P}(x \leq X \leq x + \delta) \approx f_X(x) \cdot \delta$
- $\int_{-\infty}^{\infty} f_X(x)dx = 1$

Means and Variance

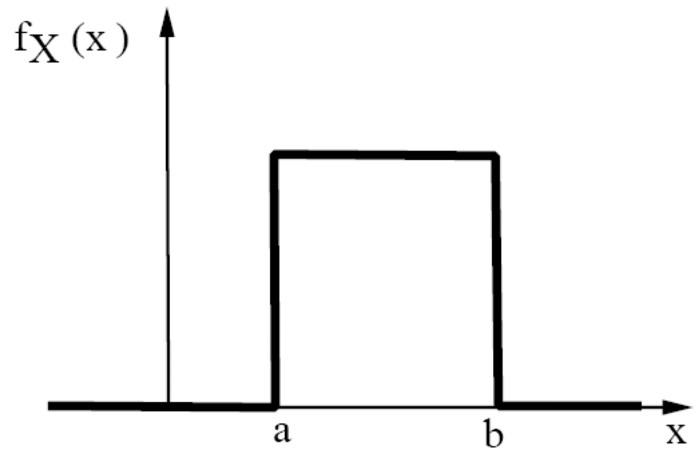
- $\mathbf{E}[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)dx$

- $\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)dx$

- $\mathsf{var}(X) = \sigma_X^2$

  $$= \int_{-\infty}^{\infty} (x - \mathbf{E}[X])^2 \cdot f_X(x)dx$$

Example: Uniform PDF



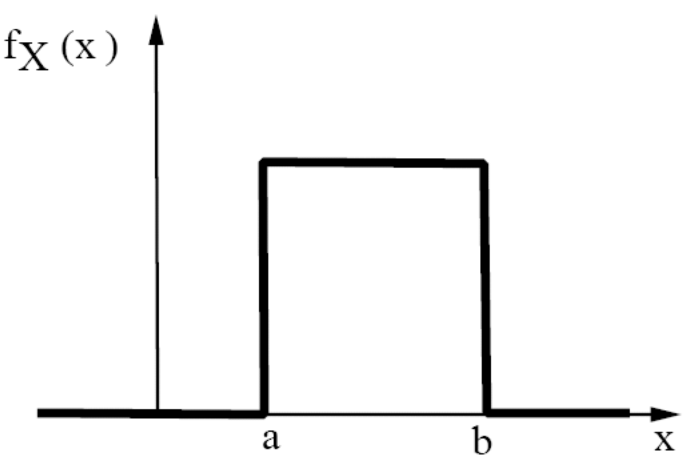
- $f_X(x) = \frac{1}{b-a} \qquad a \leq x \leq b$
- $\mathbf{E}[X] = \frac{a+b}{2}$
- $\sigma_X^2 = \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a} dx = \frac{(b-a)^2}{12}$

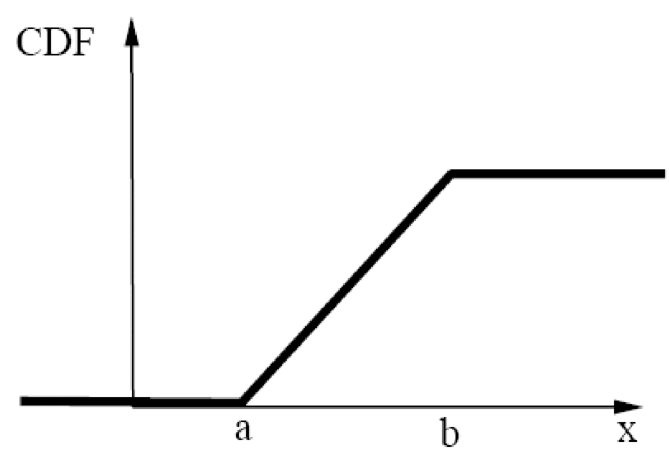
Cumulative Distribution Function

- CDF:

$$F_X(x) = \mathbf{P}(X \leq x) = \int_{-\infty}^{x} f_X(t)dt$$

- Uniform Example:

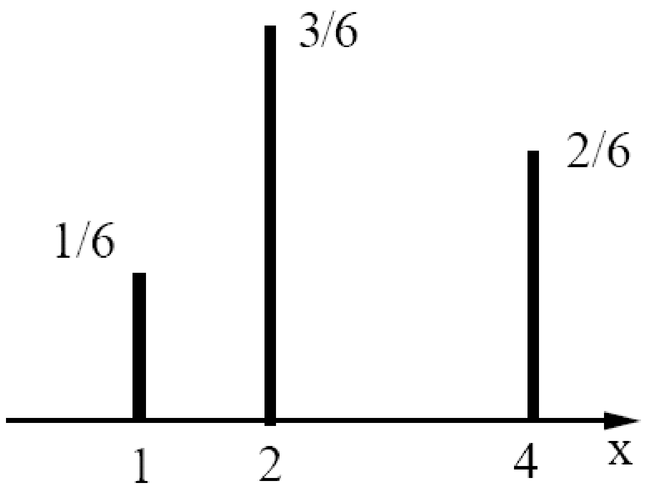


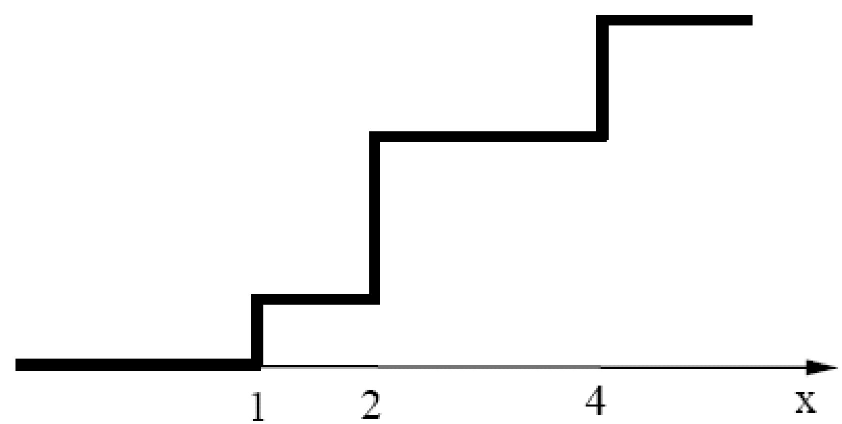


CDF for Discrete r.v.'s

$$F_X(x) = \mathbf{P}(X \leq x) = \sum_{k \leq x} p_X(k)$$

- Example:



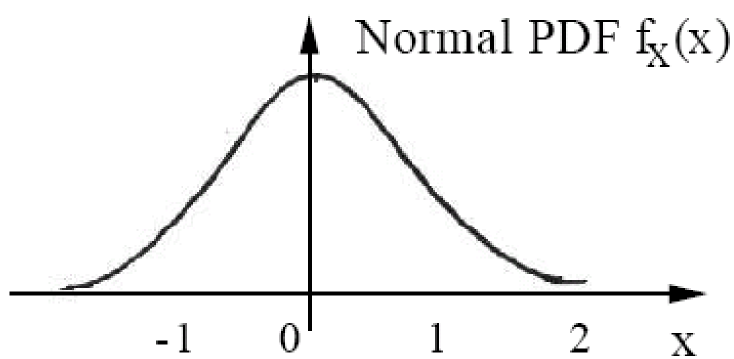


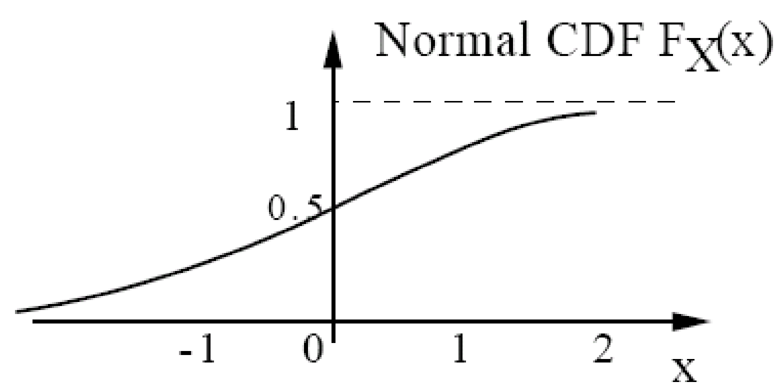
Standard Gaussian (Normal) PDF

- Standard Normal: $N(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- $\mathbf{E}[X] = 0$
- $\text{var}(X) = 1$





General Gaussian (Normal) PDF

- General Normal: $N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- It turns out that:

$\mathbf{E}[X] = \mu$ $\quad$ $\text{var}(X) = \sigma^2$

- Let $Y = aX + b$ then:

$\mathbf{E}[Y] =$ $\quad$ $\text{var}(Y) =$

- Fact: $Y \sim N(a\mu + b, a^2\sigma^2)$

Calculating Normal Probabilities

- No closed form available for CDF.
  - But, there are tables (for standard normal).

- If $X \sim N(\mu, \sigma^2)$ then $\dfrac{X-\mu}{\sigma} \sim N(\qquad\qquad)$

- If $X \sim N(2, 16)$:

$$\mathbf{P}(X \leq 3) = \mathbf{P}\left(\frac{X-2}{4} \leq \frac{3-2}{4}\right) = \text{CDF}(0.25)$$