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6.013/ESD.013J Electromagnetics and Applications, Fall 2005

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I. TM Solutions

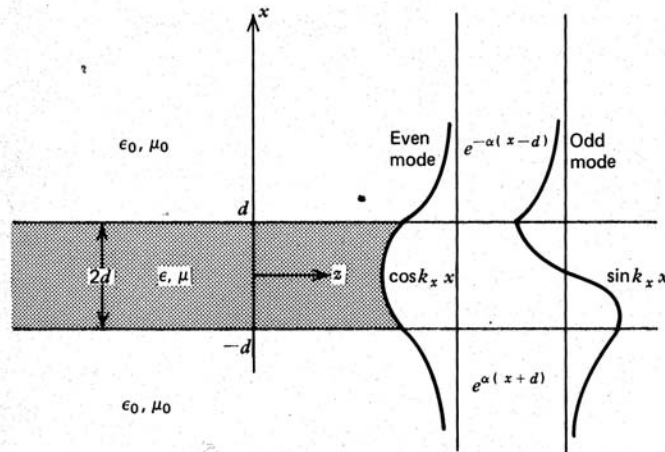


Figure 8-30 TE and TM modes can also propagate along dielectric structures. The fields can be essentially confined to the dielectric over a frequency range if the speed of the wave in the dielectric is less than that outside. It is convenient to separate the solutions into even and odd modes.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

We are considering solutions with

$$\frac{\partial}{\partial y} = 0$$

$$E_z(x, z, t) = \begin{cases} \operatorname{Re} [A_2 e^{-\alpha(x-d)} e^{j(\omega t - k_z z)}] & x \geq d \\ \operatorname{Re} [(A_1 \sin(k_x x) + B_1 \cos(k_x x)) e^{j(\omega t - k_z z)}] & |x| \leq d \\ \operatorname{Re} [A_3 e^{+\alpha(x+d)} e^{j(\omega t - k_z z)}] & x \leq -d \end{cases}$$

$$k_x^2 + k_z^2 = \omega^2 \epsilon \mu \quad (\text{in dielectric})$$

$$-\alpha^2 + k_z^2 = \omega^2 \epsilon_0 \mu_0 \quad (\text{in free space})$$

For propagation in the dielectric and evanescence in free space

$$k_z^2 = \omega^2 \epsilon \mu - k_x^2 = \omega^2 \epsilon_0 \mu_0 + \alpha^2$$

$$k_z^2 < \omega^2 \epsilon \mu, k_z^2 > \omega^2 \epsilon_0 \mu_0 \Rightarrow \omega^2 \epsilon_0 \mu_0 < k_z^2 < \omega^2 \epsilon \mu$$

A. Odd Solutions [$E_z(x, z, t) = -E_z(-x, z, t)$]

$$\hat{E}_z(x) = \begin{cases} A_2 e^{-\alpha(x-d)} & x \geq d \\ A_1 \sin(k_x x) & |x| \leq d \\ -A_2 e^{\alpha(x+d)} & x \leq -d \end{cases}$$

$$\nabla \cdot \bar{E} \Rightarrow \frac{\partial \hat{E}_x}{\partial x} - j k_z \hat{E}_z = 0$$

$$\Rightarrow \hat{E}_x = \begin{cases} -\frac{j k_z}{\alpha} A_2 e^{-\alpha(x-d)} & x > d \\ -\frac{j k_z}{k_x} A_1 \cos(k_x x) & |x| < d \\ -\frac{j k_z}{\alpha} A_2 e^{\alpha(x+d)} & x < -d \end{cases}$$

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \Rightarrow \hat{H}_x = 0, \hat{H}_z = 0$$

$$\begin{aligned} \hat{H}_y &= -\frac{1}{j\omega\mu} \left(-j k_z \hat{E}_x - \frac{\partial \hat{E}_z}{\partial x} \right) \\ &= \begin{cases} -\frac{j\omega\epsilon_0}{\alpha} A_2 e^{-\alpha(x-d)} & x \geq d \\ -\frac{j\omega\epsilon}{k_x} A_1 \cos(k_x x) & |x| \leq d \\ -\frac{j\omega\epsilon_0}{\alpha} A_2 e^{\alpha(x+d)} & x \leq -d \end{cases} \end{aligned}$$

Boundary Conditions

$$E_z(x = d_+) = E_z(x = d_-) \Rightarrow A_1 \sin(k_x d) = A_2$$

$$H_y(x = d_+) = H_y(x = d_-) \Rightarrow +\frac{j\omega\epsilon_0}{\alpha} A_2 = +\frac{j\omega\epsilon}{k_x} A_1 \cos(k_x d)$$

$$\frac{A_1}{A_2} = \frac{1}{\sin(k_x d)} = \frac{k_x \epsilon_0}{\alpha \epsilon \cos(k_x d)} \Rightarrow \alpha = \frac{\epsilon_0 k_x}{\epsilon} \tan(k_x d)$$

Critical condition for a guided wave occurs when $\alpha = 0$. At this point

$$\begin{aligned} k_x d = n\pi, k_z^2 = \omega^2 \epsilon_0 \mu_0 \\ \left(\frac{n\pi}{d}\right)^2 + \omega^2 \epsilon_0 \mu_0 = \omega^2 \epsilon \mu \Rightarrow \omega^2 = \frac{(n\pi/d)^2}{\epsilon \mu - \epsilon_0 \mu_0}, n = 1, 2, 3, \dots \end{aligned}$$

For real frequencies ($\omega^2 > 0$), $\epsilon \mu > \epsilon_0 \mu_0$

B. Even Solutions [$E_z(x, z, t) = +E_z(-x, z, t)$]

$$\hat{E}_z(x) = \begin{cases} B_2 e^{-\alpha(x-d)} & x \geq d \\ B_1 \cos(k_x x) & |x| \leq d \\ B_2 e^{\alpha(x+d)} & x \leq -d \end{cases}$$

$$\frac{\partial \hat{E}_x}{\partial x} - j k_z \hat{E}_z = 0$$

$$\Rightarrow \hat{E}_x = \begin{cases} -\frac{j k_z}{\alpha} B_2 e^{-\alpha(x-d)} & x > d \\ \frac{j k_z}{k_x} B_1 \sin(k_x x) & |x| < d \\ \frac{j k_z}{\alpha} B_2 e^{\alpha(x+d)} & x < -d \end{cases}$$

$$\hat{H}_y = \begin{cases} -\frac{j\omega\epsilon_0}{\alpha} B_2 e^{-\alpha(x-d)} & x \geq d \\ \frac{j\omega\epsilon}{k_x} B_1 \sin(k_x x) & |x| \leq d \\ \frac{j\omega\epsilon_0}{\alpha} B_2 e^{\alpha(x+d)} & x \leq -d \end{cases}$$

Boundary Conditions

$$E_z(x = d_+) = E_z(x = d_-) \Rightarrow B_2 = B_1 \cos(k_x d)$$

$$H_y(x = d_+) = H_y(x = d_-) \Rightarrow -\frac{j\omega\epsilon_0}{\alpha} B_2 = \frac{j\omega\epsilon}{k_x} B_1 \sin(k_x d)$$

$$\frac{B_2}{B_1} = \cos(k_x d) = -\frac{\epsilon\alpha}{\epsilon_0 k_x} \sin(k_x d) \Rightarrow \alpha = -\frac{\epsilon_0 k_x}{\epsilon} \cot(k_x d)$$

$$\text{Critical Condition: } \alpha = 0 \Rightarrow k_x d = (2n+1)\frac{\pi}{2}, k_z^2 = \omega^2 \epsilon_0 \mu_0$$

$$\omega^2 = \frac{\left[\frac{(2n+1)\pi}{2d}\right]^2}{\epsilon\mu - \epsilon_0\mu_0} \quad n = 0, 1, 2, \dots$$

II. TE Solutions

A. Odd Solutions

$$\hat{H}_z = \begin{cases} A_2 e^{-\alpha(x-d)} & x \geq d \\ A_1 \sin(k_x x) & |x| \leq d \\ -A_2 e^{\alpha(x+d)} & x \leq -d \end{cases}$$

$$\nabla \cdot \bar{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_z}{\partial z} \Rightarrow \frac{\partial \hat{H}_x}{\partial x} - jk_z \hat{H}_z = 0$$

$$\hat{H}_x = \begin{cases} -\frac{jk_z}{\alpha} A_2 e^{-\alpha(x-d)} & x > d \\ -\frac{jk_z}{k_x} A_1 \cos(k_x x) & |x| < d \\ -\frac{jk_z}{\alpha} A_2 e^{\alpha(x+d)} & x < -d \end{cases}$$

$$\nabla \times \bar{H} = \epsilon \frac{\partial \bar{E}}{\partial t} \Rightarrow \hat{E}_x = 0, \hat{E}_z = 0$$

$$j\omega\epsilon \hat{E}_y = -jk_z \hat{H}_x - \frac{\partial \hat{H}_z}{\partial x}$$

$$\hat{E}_y = \begin{cases} \frac{j\omega\mu_0}{\alpha} A_2 e^{-\alpha(x-d)} & x \geq d \\ \frac{j\omega\mu}{k_x} A_1 \cos(k_x x) & |x| \leq d \\ \frac{j\omega\mu_0}{\alpha} A_2 e^{\alpha(x+d)} & x \leq -d \end{cases}$$

Boundary Conditions

$$\hat{E}_y(x = d_+) = \hat{E}_y(x = d_-) \Rightarrow \frac{j\omega\mu_0}{\alpha} A_2 = \frac{j\omega\mu}{k_x} A_1 \cos(k_x d)$$

$$\hat{H}_z(x = d_+) = \hat{H}_z(x = d_-) \Rightarrow A_2 = A_1 \sin(k_x d)$$

$$\frac{A_2}{A_1} = \sin(k_x d) = \frac{\mu\alpha}{\mu_0 k_x} \cos(k_x d) \Rightarrow \alpha = \frac{\mu_0 k_x}{\mu} \tan(k_x d)$$

B. Even Solutions

$$\hat{H}_z = \begin{cases} B_2 e^{-\alpha(x-d)} & x \geq d \\ B_1 \cos(k_x x) & |x| \leq d \\ B_2 e^{\alpha(x+d)} & x \leq -d \end{cases}$$

$$\hat{H}_x = \begin{cases} -\frac{jk_z}{\alpha} B_2 e^{-\alpha(x-d)} & x > d \\ \frac{jk_z}{k_x} B_1 \sin(k_x x) & |x| < d \\ \frac{jk_z}{\alpha} B_2 e^{\alpha(x+d)} & x < -d \end{cases}$$

$$\hat{E}_y = \begin{cases} \frac{j\omega\mu_0}{\alpha} B_2 e^{-\alpha(x-d)} & x \geq d \\ -\frac{j\omega\mu}{k_x} B_1 \sin(k_x x) & |x| \leq d \\ -\frac{j\omega\mu_0}{\alpha} B_2 e^{\alpha(x+d)} & x \leq -d \end{cases}$$

$$\hat{E}_y(x = d_+) = \hat{E}_y(x = d_-) \Rightarrow \frac{j\omega\mu_0}{\alpha} B_2 = -\frac{j\omega\mu}{k_x} B_1 \sin(k_x d)$$

$$\hat{H}_z(x = d_+) = \hat{H}_z(x = d_-) \Rightarrow B_2 = B_1 \cos(k_x d)$$

$$\frac{B_2}{B_1} = \cos(k_x d) = -\frac{\alpha\mu}{\mu_0 k_x} \sin(k_x d) \Rightarrow \alpha = -\frac{\mu_0 k_x}{\mu} \cot(k_x d)$$