Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.011: Signals, Systems and Inference

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Problem Set 7
WSS Processes

Reading: Chapter 10

Some preliminaries: We've introduced the mean function $\mu_X(t) = E[X(t)]$ and autocorrelation function $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$ of a random process $X(t)$ in last Wednesday's lecture, and the notion of wide-sense (or weak-sense) stationarity, WSS, in last Thursday's recitation. For a WSS process, the mean function does not depend on time, so $\mu_X(t) = \mu_X$, and the autocorrelation function depends only on the **lag** $\tau = t_2 - t_1$ rather than on t_1 and t_2 individually, so $R_{XX}(t + \tau, t) = R_{XX}(\tau, 0)$ for all t. We can therefore streamline the notation for the autocorrelation function of a WSS process $X(t)$ to $R_{XX}(\tau)$. Note also that the auto*covariance* function $C_{XX}(t_1, t_2)$ can be computed as $R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$; for a WSS process this becomes $C_{XX}(\tau) = R_{XX}(\tau) - \mu_X^2$.

Problem 7.1

A waveform used in a digital communication system is modeled as a random signal (or process) $X(t)$ with the following properties:

- The signal is piecewise constant over intervals (or "slots") of length T, and has the value A_k in the kth interval, where the values $\{A_k\}$ form an i.i.d. sequence with mean 0 and variance σ_A^2 for each k.
- The start time of the first interval following the time origin $t = 0$ is equally likely to be any value between 0 and T. Thus $X(t) = A_1$ for $D < t < D + T$, where D is *uniformly distributed* in the interval $0 < D < T$. (This is to model the situation where the transmitter and receiver clocks are not synchronized, so the actual start of the first slot is not known at the receiver.)

(a) Draw a labeled sketch of a typical realization of this process, i.e., a typical signal, to help you visualize what's going on.

(b) What is $\mu_X(t) = E[X(t)]$?

(c) What is $R_{XX}(t_1, t_2)\Big(= R_{XX}(t_2, t_1)\Big)$ when $|t_1 - t_2| > T$?

(d) Choose t_1 and t_2 such that $0 < t_1 < t_2 < T$, and determine $R_{XX}(t_1, t_2)$. (Your analysis will probably be helped by separately considering two cases: first where $0 < D < t_1$ or $t_2 < D < T$, and second where $t_1 < D < t_2$.)

(e) Write $R_{XX}(t_1 + \ell T, t_2 + \ell T)$ in terms of $R_{XX}(t_1, t_2)$ for an arbitrary integer ℓ.

(f) Putting together all the above parts, you can conclude that the process is WSS. What is $R_{XX}(\tau)$, where $\tau = t_1 - t_2$?

(g) Did you need the (zero-mean) A_k and A_i for $k \neq i$ to be *independent* for your analysis above to work, or would it have sufficed to have them be *uncorrelated*?

Problem 7.2

Consider a DT random process $X[n]$ defined as follows: the values at distinct times are chosen independently; for n even, $X[n]$ is $+1$ or -1 with equal probability; for n odd, $X[n] = \frac{1}{3}$ with probability $\frac{9}{10}$ and $X[n] = -3$ with probability $\frac{1}{10}$. The process is clearly not strict-sense stationary. Determine the mean and autocorrelation functions of the process, and thus decide whether the process is wide-sense stationary.

Problem 7.3

Problem 10.34 (which is Problem 10.38 in the softcover version).

Problem 7.4

This problem develops the condition for a DT WSS process to be ergodic in mean value (see definition below). The CT case is developed in Problem 10.43 of SSI, which is Problem 10.36 in the softcover version, if you wish to look there for connections or inspiration.

Consider the random variable Y obtained by time-averaging a WSS process $X[\cdot]$ over the interval $[-L, L]$:

$$Y = \frac{1}{2L+1} \sum_{n=-L}^{L} X[n] \; .$$

This quantity is the (local) *time-average* of the process $X[\cdot]$ (where "local" refers to the fact that the average is taken in the vicinity of time 0 — but our eventual interest will be in the case $L \nearrow \infty$). For any particular realization $x[\cdot]$ of the process, the random variable Y takes the value

$$y = \frac{1}{2L+1} \sum_{n=-L}^{L} x[n] \; .$$

(We will usually be less fussy about distinguishing notationally between a random variable and a particular realized value of it, but it is worth doing in the present context.) The values y that Y takes in different experiments (or realizations) will be centered around the mean value $E[Y] = \mu_Y$ of Y, with a spread that is indicated by the standard deviation σ_Y of Y.

(a) Show that $E[Y] = \mu_Y = \mu_X$, so the expected value of the *time*-average of the random process $X[\cdot]$ is the *ensemble*-average of the process.

(b) Find an explicit expression for the variance σ_Y^2 of Y in terms of the auto*covariance* function $C_{XX}[m]$ of $X[\cdot]$. (To keep your calculations clean, your first step should be to relate the deviation $\widetilde{Y} = Y - \mu_Y$ to the relevant deviations $\widetilde{X}[n] = X[n] - \mu_X$.)

Check/hint: You should be able to write your expression for σ_Y^2 in the form

$$\frac{1}{(K+1)} \sum_{m=-K}^{K} \Big(1 - \frac{|m|}{K+1}\Big) C_{XX}[m]$$

for an appropriately chosen K – what K?

A WSS process $X[\cdot]$ for which $\sigma_Y^2 \searrow 0$ as $L \nearrow \infty$ has the property that time-averages computed in different experiments cluster more and more closely (as $L \nearrow \infty$) around the ensemble mean $\mu_Y = \mu_X$, because the variance of this time-average tends to 0. We say in such a case, where the time-average of a WSS process $X[\cdot]$ tends to its ensemble mean, that the process is *ergodic* in mean value. This is a significant generalization of the Weak Law of Large Numbers beyond the case of processes that are uncorrelated across time, to the more general case of WSS processes. So it is of interest to know what conditions on the original process $X[\cdot]$ will guarantee that $\sigma_Y^2 \searrow 0$ as $L \nearrow \infty$. This motivates the following questions.

(c) For each of the following processes in (i)-(iii), use the expression you derived in (b) to determine whether $\sigma_Y^2 \searrow 0$ as $L \nearrow \infty$. Note that it is not necessary to actually evaluate your expression in (b) exactly for the following cases; it could suffice, depending on what you are trying to establish, to determine an upper bound on σ_Y^2 that tends to 0 as $L \nearrow \infty$, or determine a lower bound on σ_Y^2 that tends to a positive value as $L \nearrow \infty$.

   (i) $C_{XX}[m]$ nonzero over only a finite range of m, say $|m| \leq M$, and 0 everywhere else. A particular case of this is an iid process, where $C_{XX}[m] = \sigma_X^2 \delta[m]$. (Note that iid implies the autocovariance is nonzero only at $m = 0$, but the converse is not true: having an autocovariance of this form doesn't mean that samples at different times are independent, only that they are uncorrelated.)

   (ii) $C_{XX}[m] = 3\alpha^{|m|}$ for some positive or negative α satisfying $|\alpha| < 1$.

   (iii) $C_{XX}[m] = (0.8)^{|m|} + 2$.

Some subtler cases that we don't ask you to solve here, but for which it turns out the σ_Y^2 you computed in (b) tends to 0 as $L \nearrow \infty$, are the following:

$$\text{(iv)} \quad C_{XX}[m] \searrow 0 \quad \text{as} \quad |m| \nearrow \infty\ ,$$

and

$$\text{(v)} \quad C_{XX}[m] = \cos(\Omega_0 m)\ .$$

Note that (i) and (ii) above are special cases of (iv).

The sufficient condition (iv) is a **good one to remember**: Any WSS process $X[\cdot]$ whose autocovariance function $C_{XX}[m]$ tends to 0 as $|m|$ tends to ∞ is ergodic in mean value. This condition, though sufficient, is not necessary — as case (v) shows; in case (v) the autocovariance function does not tend to any limit as $|m| \nearrow \infty$ but the process is still ergodic in mean value (because σ_Y^2 from (b) does in fact tend to 0 as $L \nearrow \infty$).