Fresnel Equations and Light Guiding

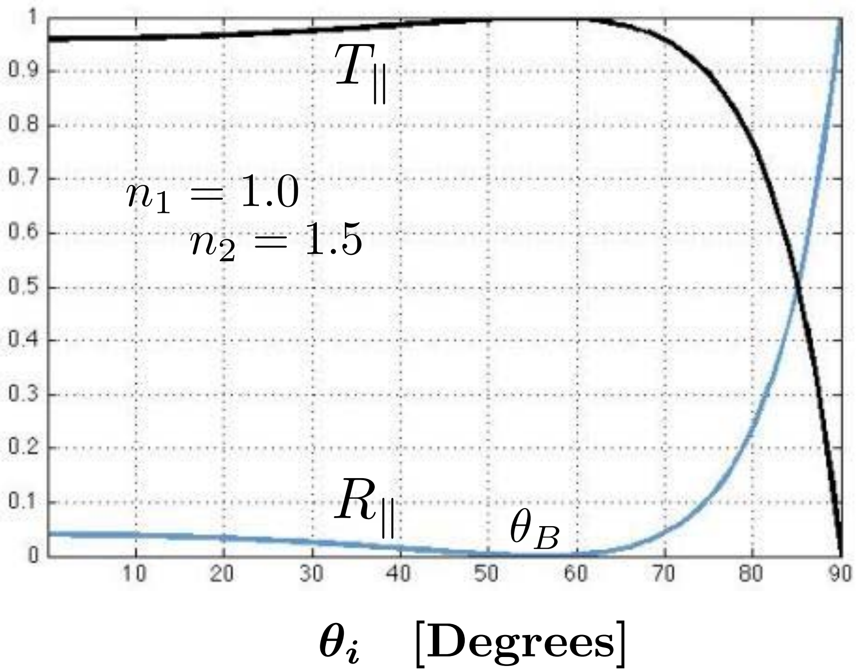
Reading - Shen and Kong – Ch. 4

Outline

- Review of Oblique Incidence
- Review of Snell's Law
- Fresnel Equations
- Evanescence and TIR
- Brewster's Angle
- EM Power Flow

TRUE / FALSE

1. The Fresnel equations describe reflection and transmission coefficients as a function of intensity. ____

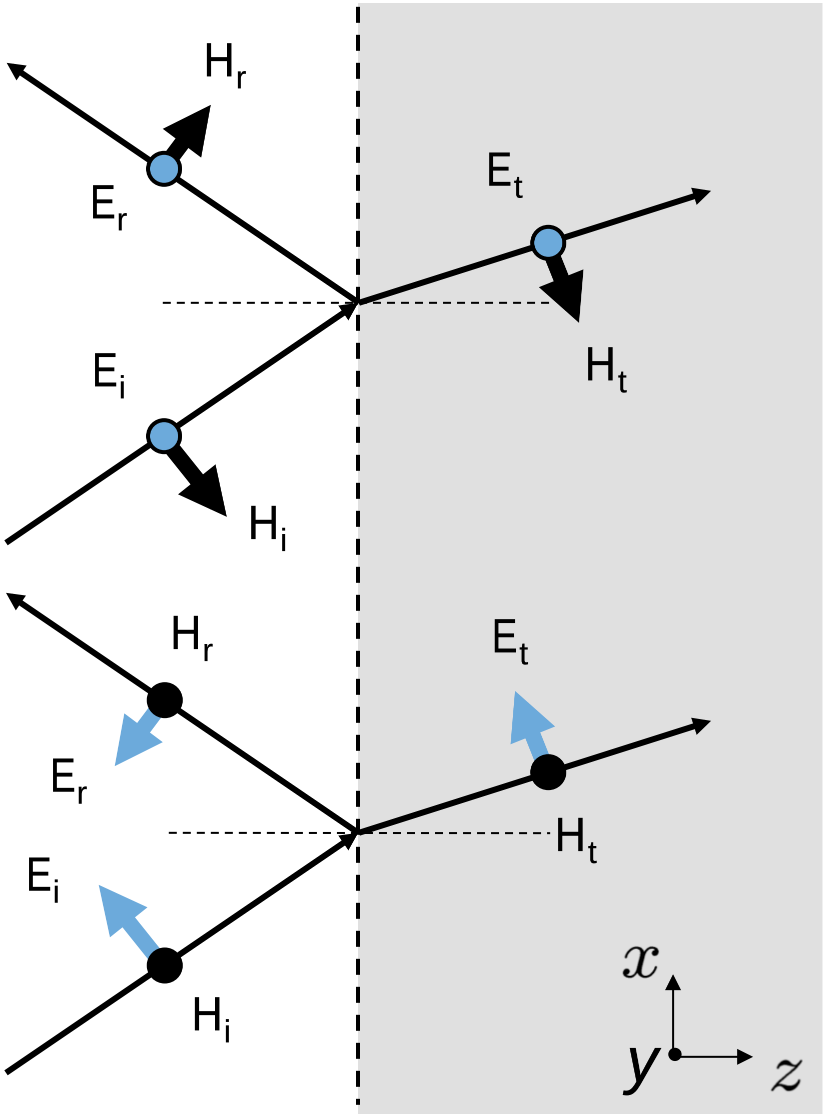


2. This is the power reflection and transmission plot for an EM wave that is TE (transverse electric) polarized:

____

3. The phase matching condition for refraction is a direct result of the boundary conditions. ____

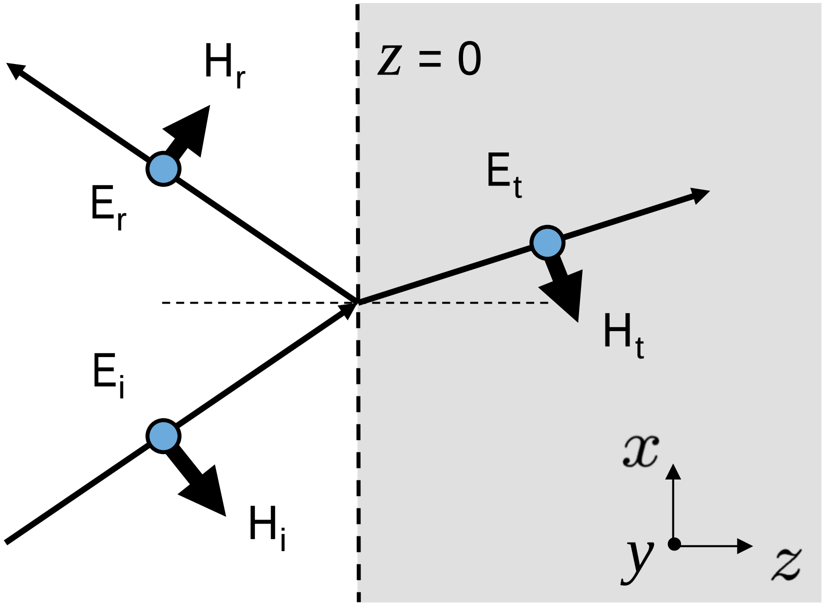
Oblique Incidence at Dielectric Interface



Transverse
Electric Field

Transverse
Magnetic Field

Partial TE Analysis



$$\vec{E}_i = \hat{y} E_o^i e^{-jk_{ix}x - jk_{iz}z}$$

$$\vec{E}_r = \hat{y} E_o^r e^{-jk_{rx}x + jk_{rz}z}$$

$$\vec{E}_t = \hat{y} E_o^t e^{-jk_{tx}x - jk_{tz}z}$$

$$\omega_i = \omega_r = \omega_t$$

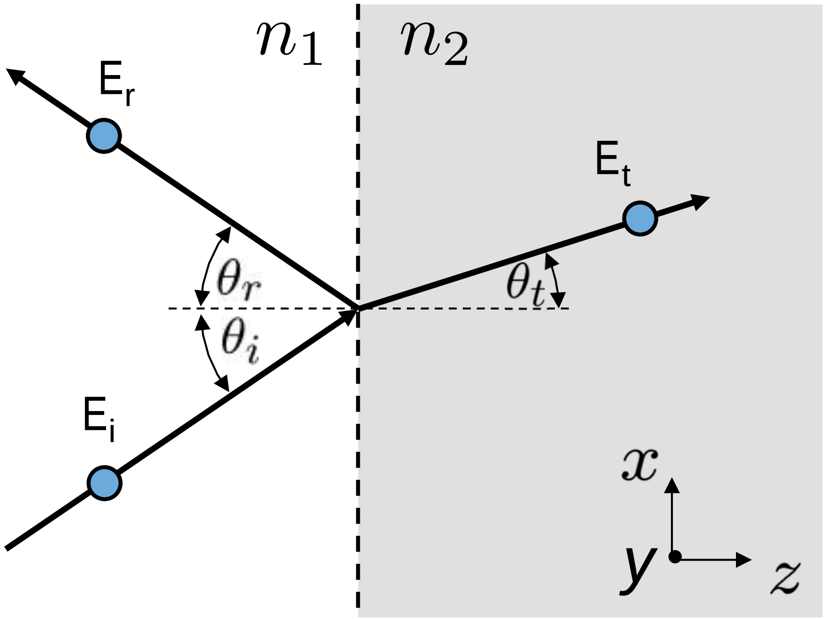
Tangential E must be continuous at the boundary z = 0 for all x and for t.

$$E_o^i e^{-jk_{ix}x} + E_o^r e^{-jk_{rx}x} = E_o^t e^{-jk_{tx}x}$$

This is possible if and only if $k_{ix} = k_{rx} = k_{tx}$ and $\omega_i = \omega_r = \omega_t$.

The former condition is phase matching $k_{ix} = k_{rx} = k_{tx}$

Snell's Law



$$k_{ix} = k_{rx}$$

$$k_{ix} = k_{tx}$$

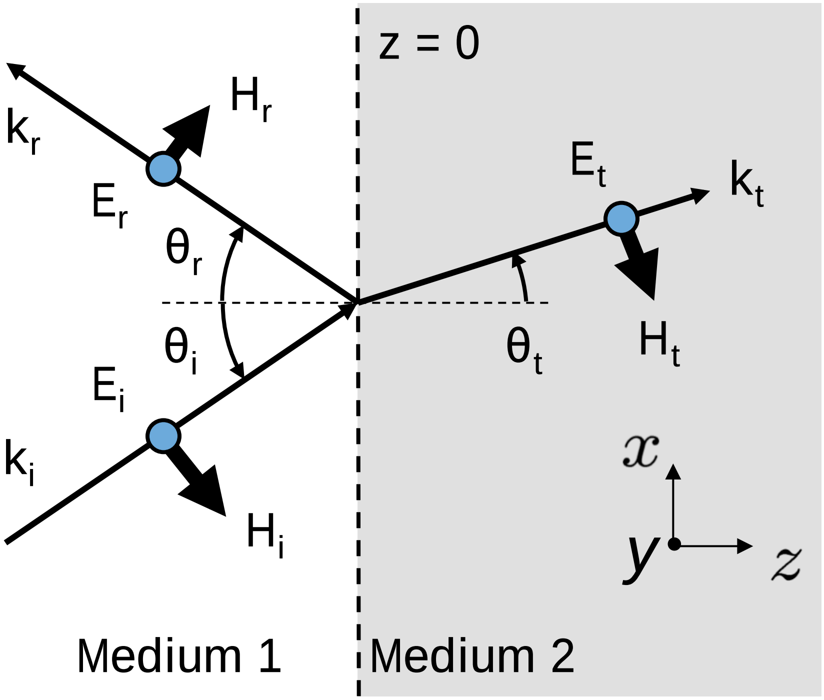
$$n_1 \sin\theta_i = n_1 \sin\theta_r$$

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

$$\theta_i = \theta_r$$

Snell's Law

TE Analysis - Set Up



$$\vec{E}_i = \hat{y}E_o e^{j(-k_{ix}x - k_{iz}z)}$$

$$\vec{E}_r = \hat{y}rE_o e^{j(-k_{ix}x + k_{iz}z)}$$

$$\vec{E}_t = \hat{y}tE_o e^{j(-k_{ix}x - k_{iz}z)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \mu \vec{H}}{\partial t}$$

$$-jk \times \vec{E} = -j\omega\mu\vec{H}$$

$$\vec{H} = \frac{1}{\omega\mu} k \times \vec{E}$$

To get H, use Faraday's Law

$$\vec{H}_i = (\hat{z}k_{ix} - \hat{x}k_{iz})\frac{E_o}{\omega\mu_1}e^{j(-k_{ix}x - k_{iz}z)}$$

$$\vec{H}_r = (\hat{z}k_{ix} + \hat{x}k_{iz})\frac{rE_o}{\omega\mu_1}e^{j(-k_{ix}x + k_{iz}z)}$$

$$\vec{H}_t = (\hat{z}k_{tx} + \hat{x}k_{tz})\frac{tE_o}{\omega\mu_2}e^{j(-k_{tx}x - k_{tz}z)}$$

$$k_x^2 + k_z^2 = k^2 = \omega^2\mu\epsilon$$

$$k_x = k\sin\theta$$

$$k_z = k\cos\theta$$

TE & TM Analysis – Solution

TE solution comes directly from the boundary condition analysis

$$r = \frac{\eta_t \cos\theta_i - \eta_i \cos\theta_t}{\eta_t \cos\theta_i + \eta_i \cos\theta_t} \qquad t = \frac{2\eta_t \cos\theta_i}{\eta_t \cos\theta_i + \eta_i \cos\theta_t}$$

TM solution comes from ε ↔ μ

$$r = \frac{\eta_t^{-1} \cos\theta_i - \eta_i^{-1} \cos\theta_t}{\eta_t^{-1} \cos\theta_i + \eta_i^{-1} \cos\theta_t} \qquad t = \frac{2\eta_t^{-1} \cos\theta_i}{\eta_t^{-1} \cos\theta_i + \eta_i^{-1} \cos\theta_t}$$

Note that the TM solution provides the reflection and transmission coefficients for H, since TM is the dual of TE.

Fresnel Equations - Summary

From Shen and Kong … just another way of writing the same results

TE Polarization

$$r_{\mathrm{TE}} = \frac{E_o^r}{E_o^i} = \frac{\mu_2 k_{iz} - \mu_1 k_{tz}}{\mu_2 k_{iz} + \mu_1 k_{tz}}$$

$$t_{\mathrm{TE}} = \frac{E_o^t}{E_o^i} = \frac{2\mu_2 k_{iz}}{\mu_2 k_{iz} + \mu_1 k_{tz}}$$

TM Polarization

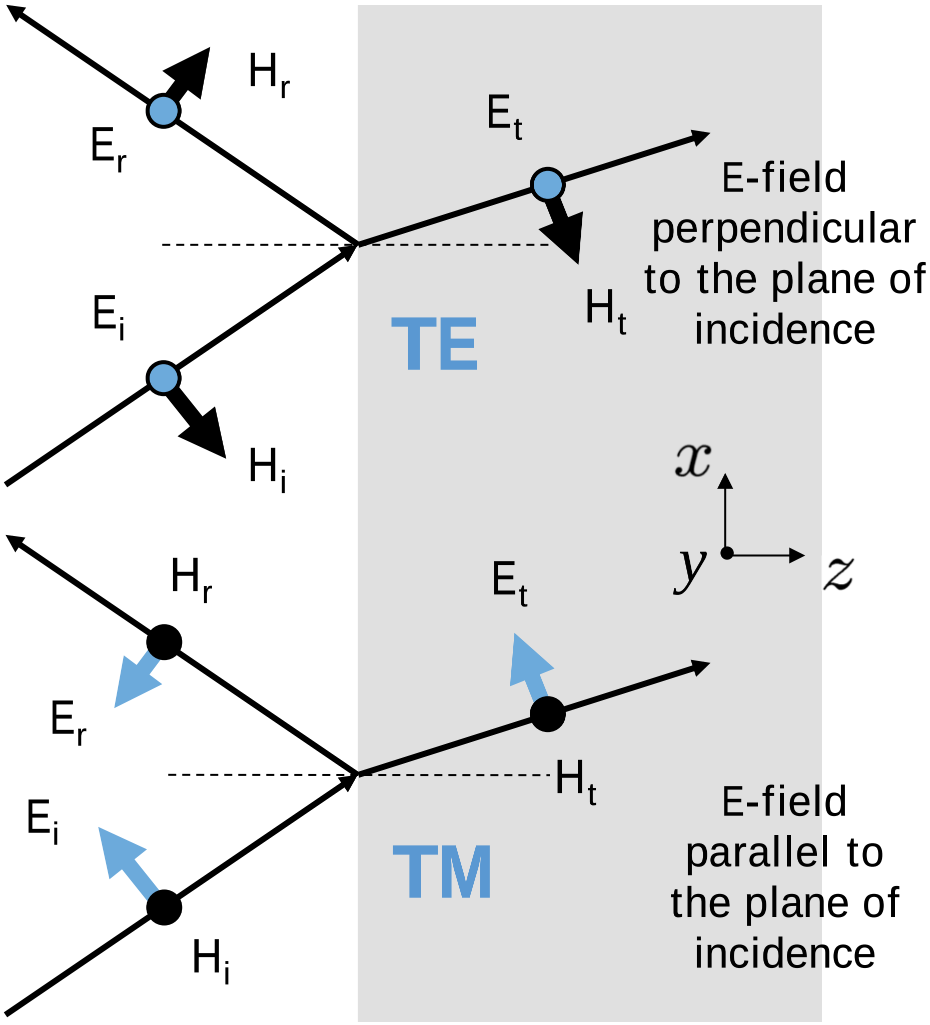
$$r_{\mathrm{TM}} = \frac{E_o^r}{E_o^i} = \frac{\epsilon_2 k_{iz} - \epsilon_1 k_{tz}}{\epsilon_2 k_{iz} + \epsilon_1 k_{tz}}$$

$$t_{\mathrm{TM}} = \frac{E_o^t}{E_o^i} = \frac{2\epsilon_2 k_{iz}}{\epsilon_2 k_{iz} + \epsilon_1 k_{tz}}$$

Reflection of Light (Optics Viewpoint … $\mu_1 = \mu_2$)

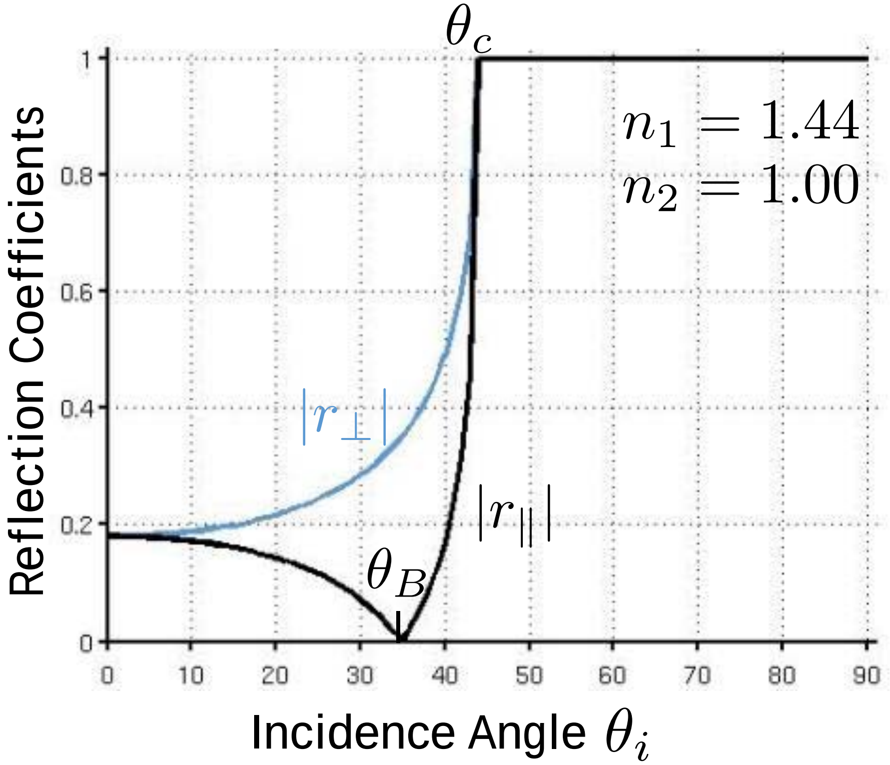
TE: $$r_\perp = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$

TM: $$r_\parallel = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t}$$



E-field perpendicular to the plane of incidence

E-field parallel to the plane of incidence



Brewster's Angle

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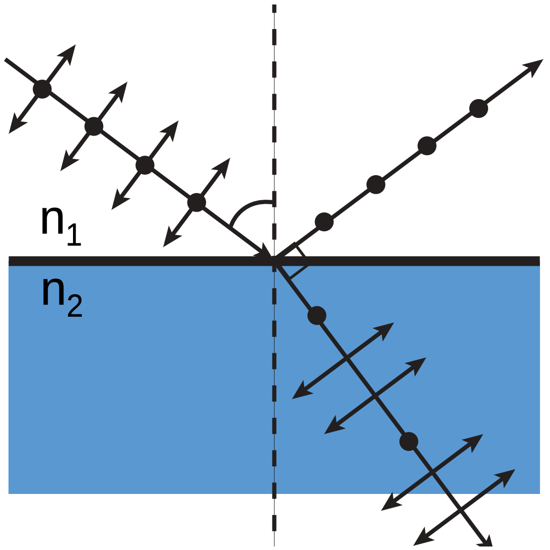
Sir David Brewster (1781 –1868) was a Scottish scientist, inventor and writer. Rediscovered and popularized kaleidoscope in 1815.



Incident ray (unpolarised)

Reflected ray (TE polarised)

$\theta_B + \theta_t = 90°$

Refracted ray (slightly polarised)

$$n_1 \sin\left(\theta_B\right) = n_2 \sin\left(90 - \theta_B\right)$$
$$= n_2 \cos\left(\theta_B\right)$$

$$\boxed{\theta_B = \arctan\left(\frac{n_2}{n_1}\right)}$$

<u>@ $\theta_i = \theta_B$</u>

TE: $$r_\perp = \frac{n_1\cos\theta_B - n_2\cos\theta_t}{n_1\cos\theta_B + n_2\cos\theta_t} \neq 0$$

TM: $$r_\parallel = \frac{n_2\cos\theta_B - n_1\cos\theta_t}{n_2\cos\theta_B + n_1\cos\theta_t} = 0$$

$$\Rightarrow \quad \left.\begin{aligned} n_2\cos\theta_B &= n_1\cos\theta_t \\ n_1\sin\theta_B &= n_2\sin\theta_t \end{aligned}\right\}$$

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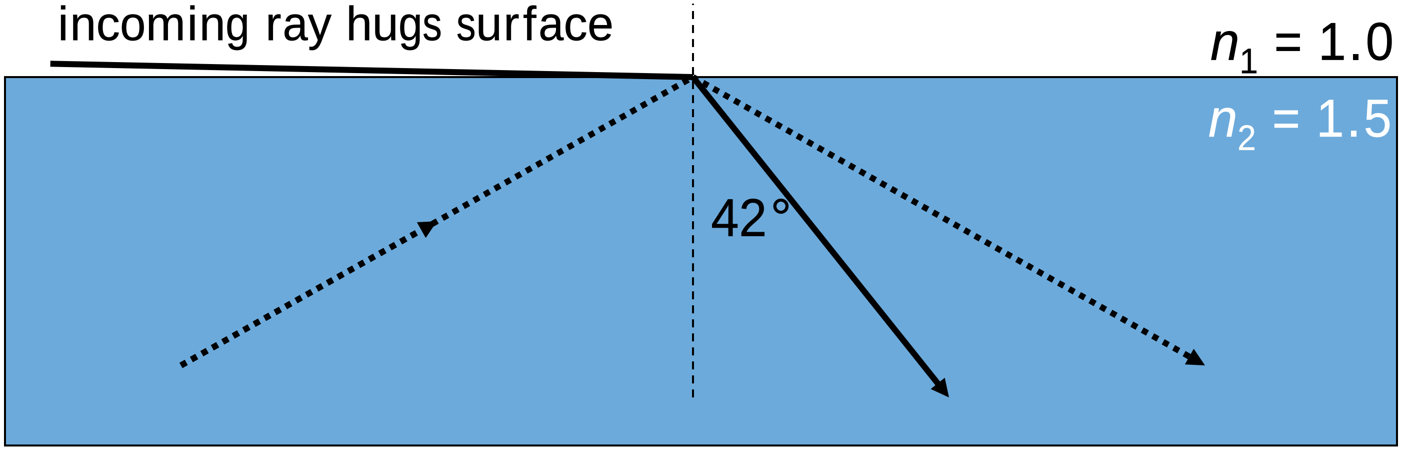
Total Internal Reflection

Beyond the critical angle,
refraction no longer occurs

- thereafter, you get *total internal reflection*

$$n_2 \sin\theta_2 = n_1 \sin\theta_1 \rightarrow \theta_{crit} = \sin^{-1}(n_1 / n_2)$$

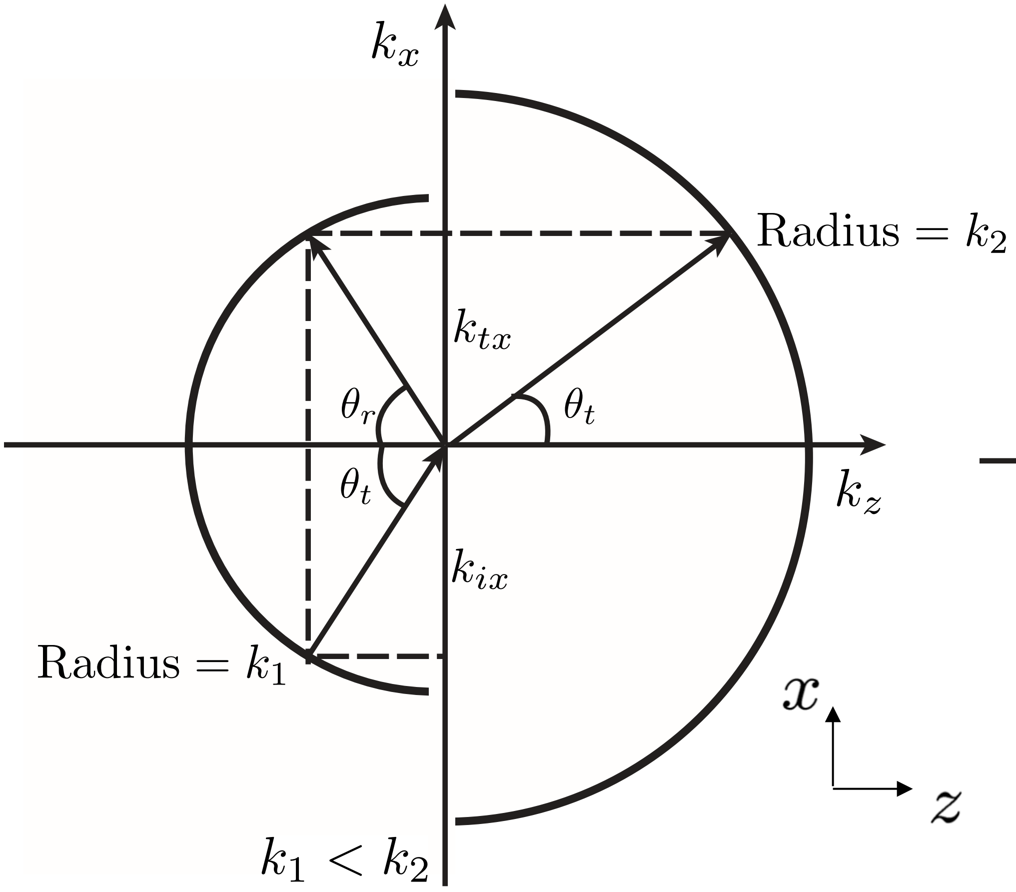
- for glass ($n_2 = 1.5$), the critical internal angle is 42°
- for water, it's 49°
- a ray within the higher index medium cannot escape at shallower angles (look at sky from underwater…)



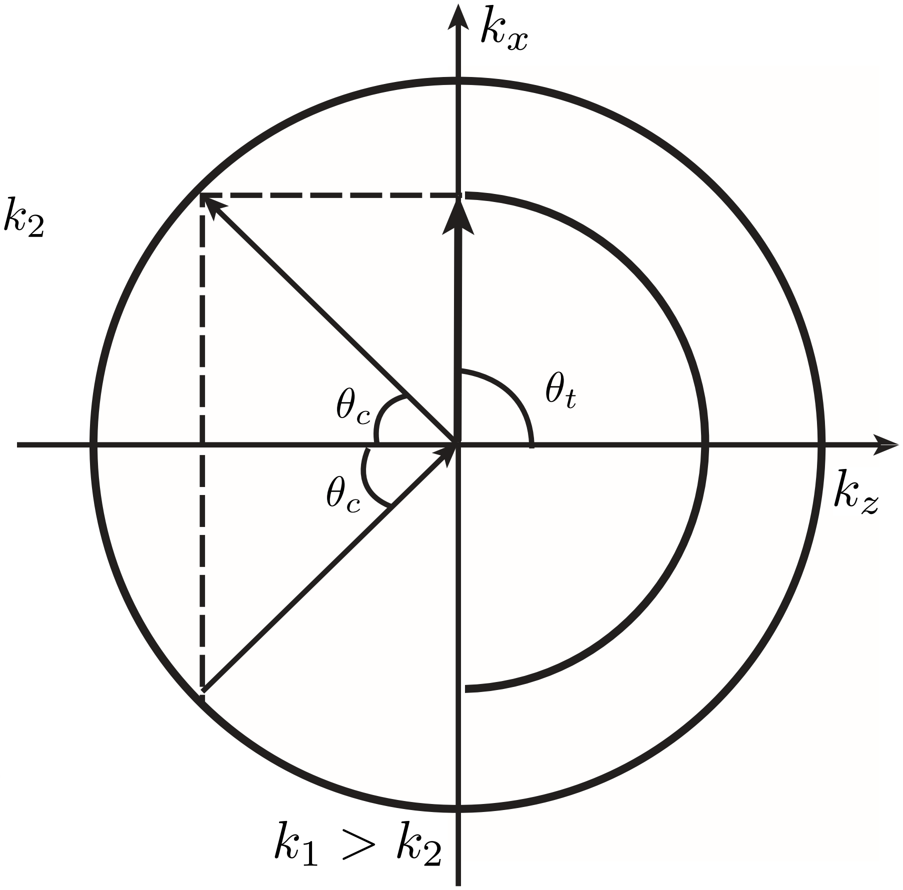
Snell's Law Diagram

Tangential field is continuous … $k_{ix} = k_{it}$

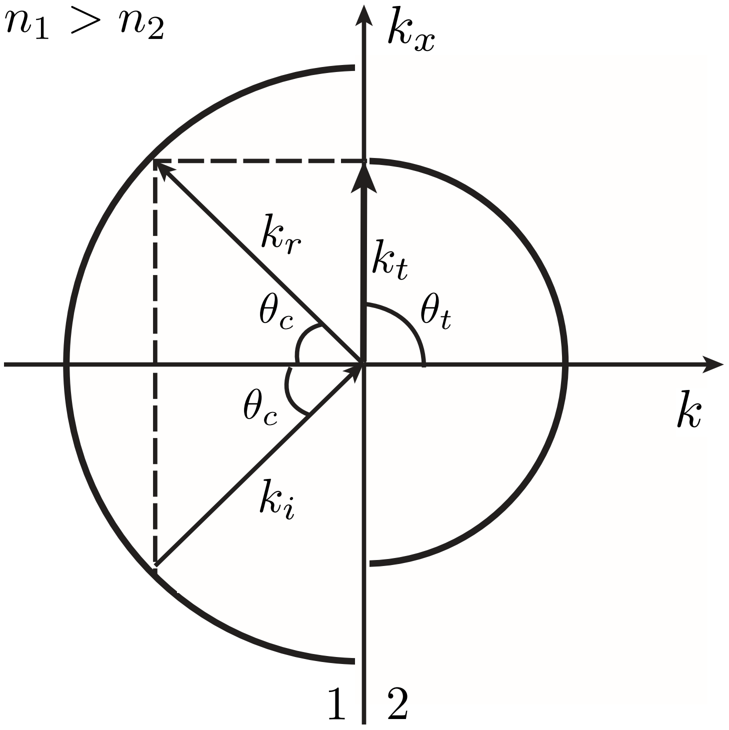
Refraction



Total Internal Reflection



Total Internal Reflection & Evanscence



Snell's Law dictates $n_1 \sin(\theta_i) = n_2 \sin(\theta_t)$, or equivalently, $k_{ix} = k_{tx}$. For $n_1 > n_2$, $\theta_t = 90°$ at $\theta_i = \sin^{-1}(n_2/n_1) \equiv \theta_C$. What happens for $\theta_i > \theta_C$?

$k_{tz}^2 = k_t^2 - k_{tx}^2 < 0 \rightarrow k_{tz} = \pm j \alpha_{tz}$, with α_{tz} real.

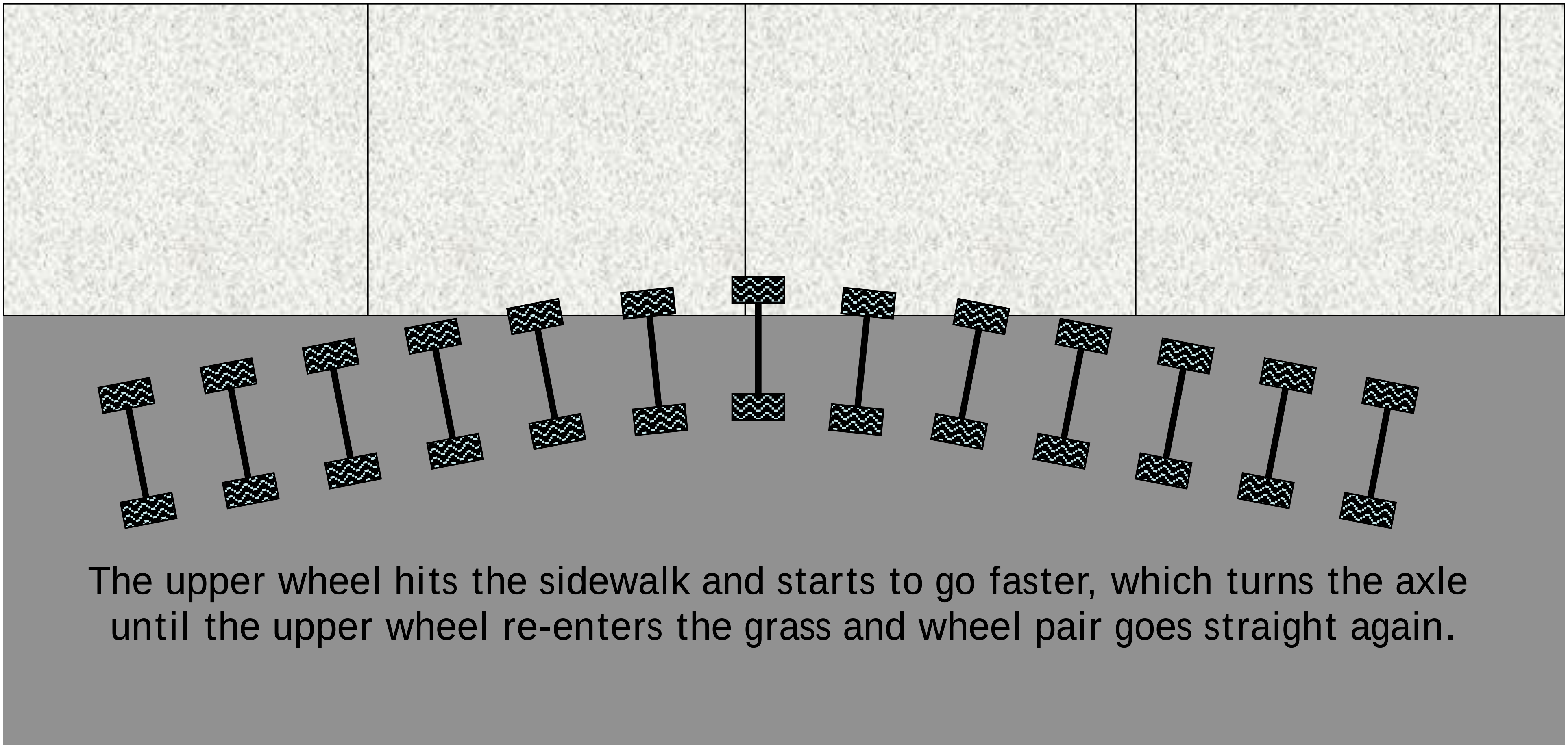
The refracted, or transmitted, wave takes the complex exponential form

$\exp(- j k_{tx} x - \alpha_{tz} z)$.

This is a non-uniform plane wave that travels in the x direction and decays in the z direction. It carries no time average power into Medium 2. This phenomenon is referred to as total internal reflection. This is the similar to reflection of radio waves by the ionosphere.

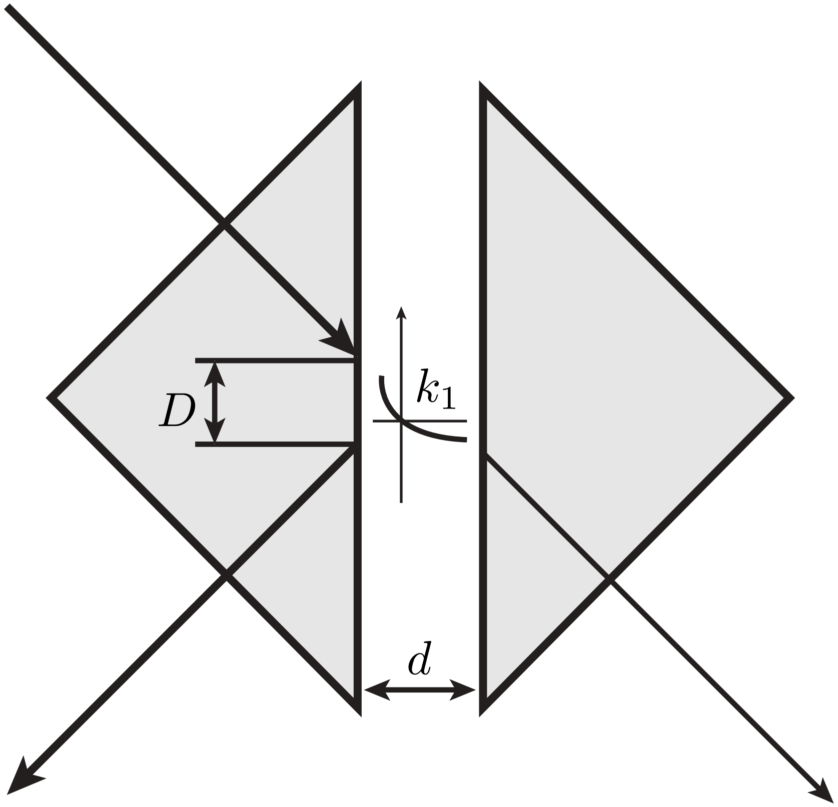
Total Internal Reflection in Suburbia

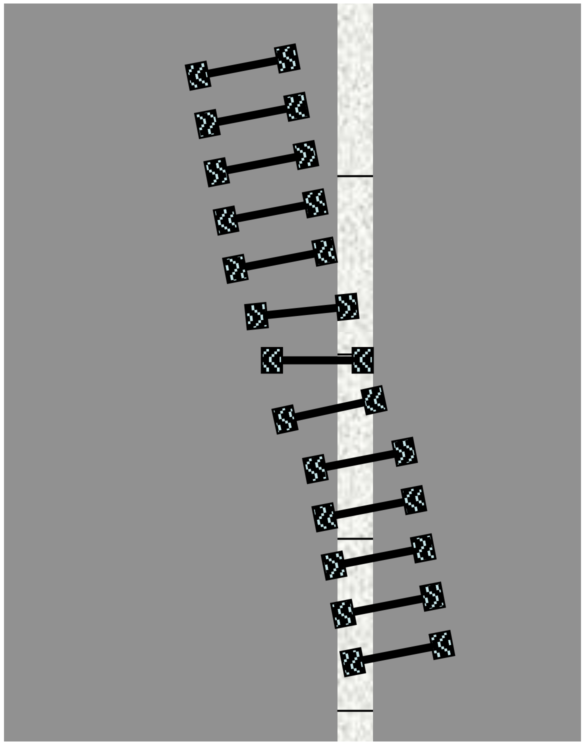
Moreover, this wheel analogy is mathematically equivalent to the refraction phenomenon. One can recover Snell's law from it: $n_1\sin\theta_1 = n_2\sin\theta_2$.



The upper wheel hits the sidewalk and starts to go faster, which turns the axle until the upper wheel re-enters the grass and wheel pair goes straight again.

Frustrated Total Internal Reflection In Suburbia





An evanescent field can propagate once the
field is again in a high-index material.

Applications of Evanescent Waves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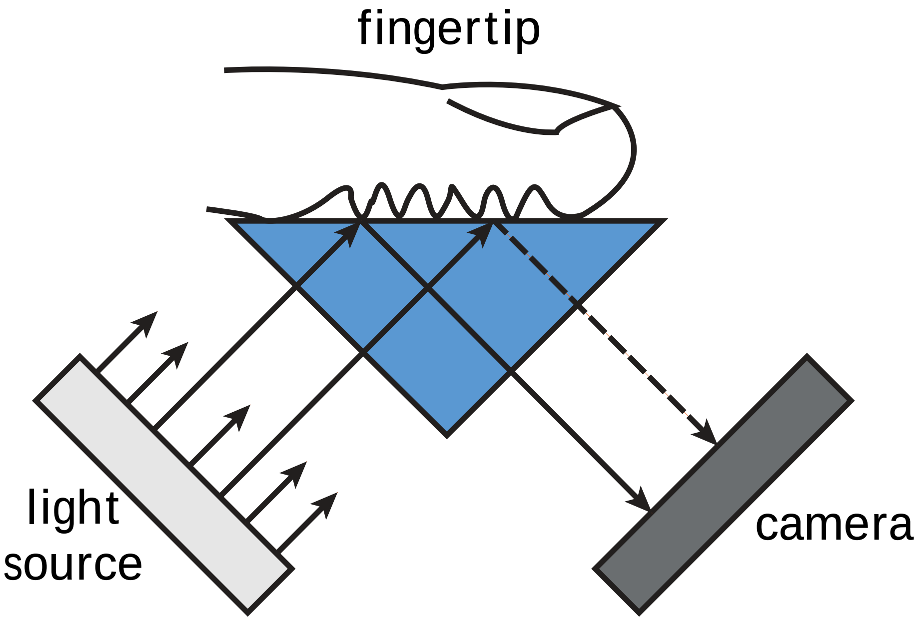


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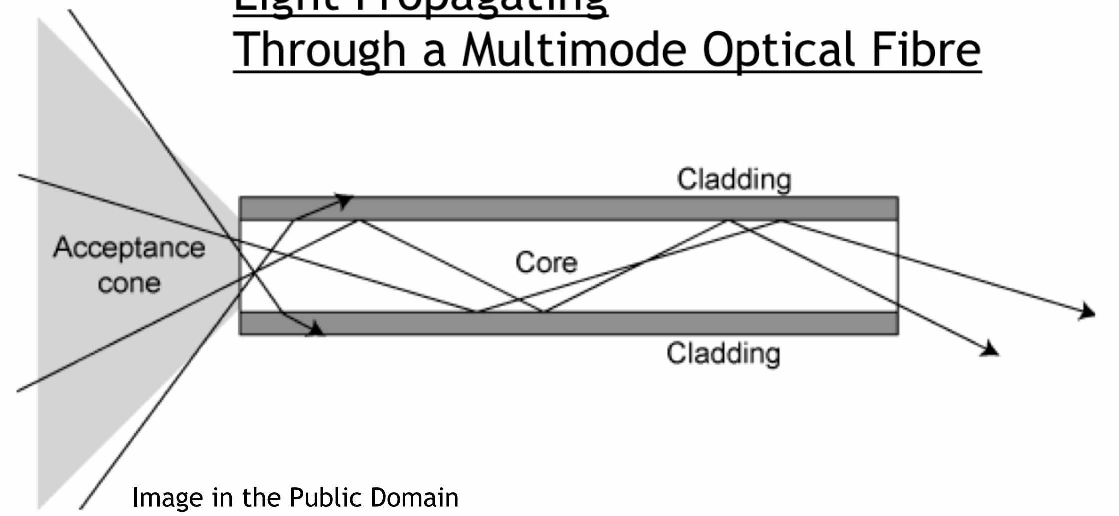
The camera observes *TIR* from a fingerprint valley and blurred *TIR* from a fingerprint ridge.

Light Propagating Through a Multimode Optical Fibre



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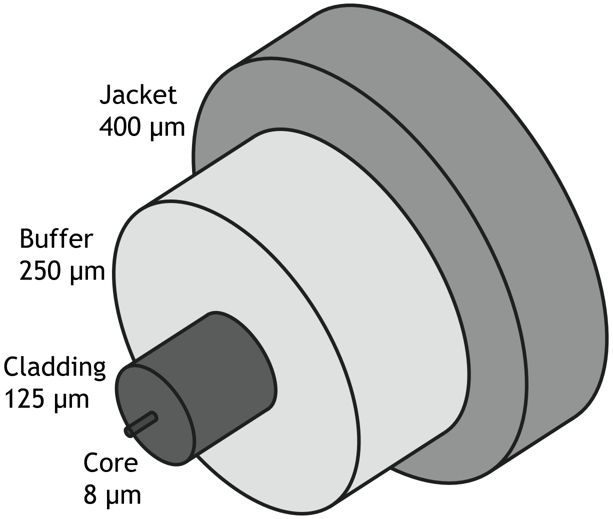

Single Mode Fibre Structure

The optic fiber used in undersea cables is chosen for its exceptional clarity, permitting runs of more than 100 kilometers between repeaters to minimize the number of amplifiers and the distortion they cause.

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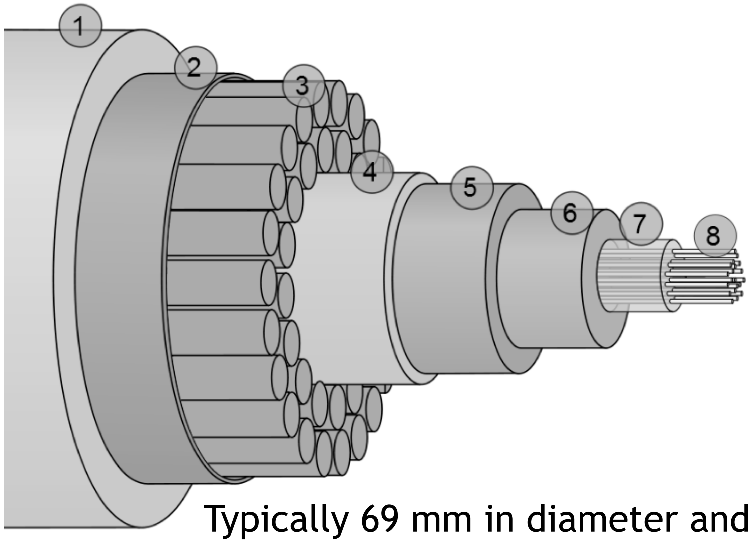

A cross-section of a communications cable:

1. Polyethylene
2. "Mylar" tape
3. Stranded steel wires
4. Aluminum water barrier
5. Polycarbonate
6. Copper or aluminum tube
7. Petroleum jelly
8. Optical fibers

Typically 69 mm in diameter and weigh around 10 kg per meter

Submarine communication cables crossing the Scottish shore

Image by Jmb at http://en.wikipedia.org/wiki/File:Submarine_Telephone_Cables_PICT8182_1.JPG on Wikipedia.

Optical Waveguides Examples

Image by Apreche
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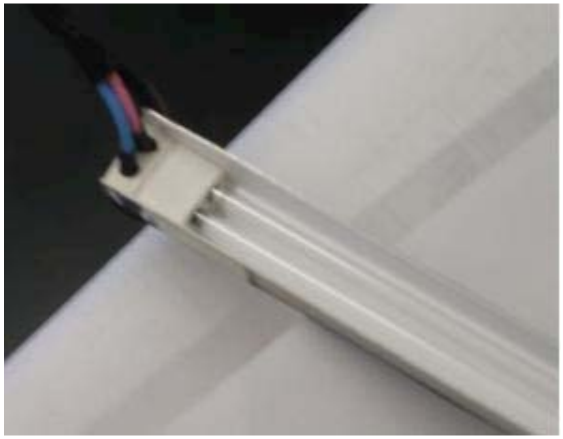

Image by Rberteig
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LCD screen lit by two backlights coupled into a flat waveguide

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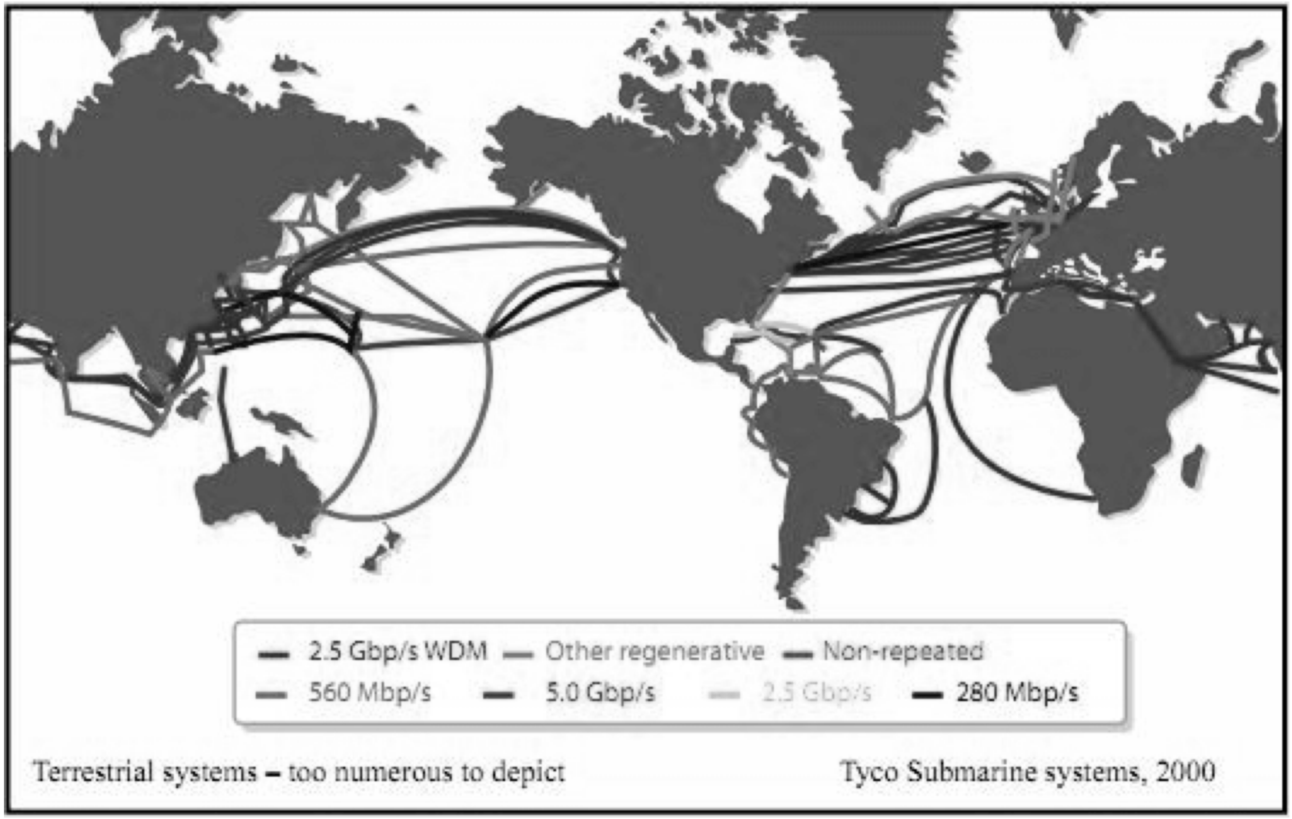
Optical fiber

Today’s Culture Moment

Global Fiber Optic Network



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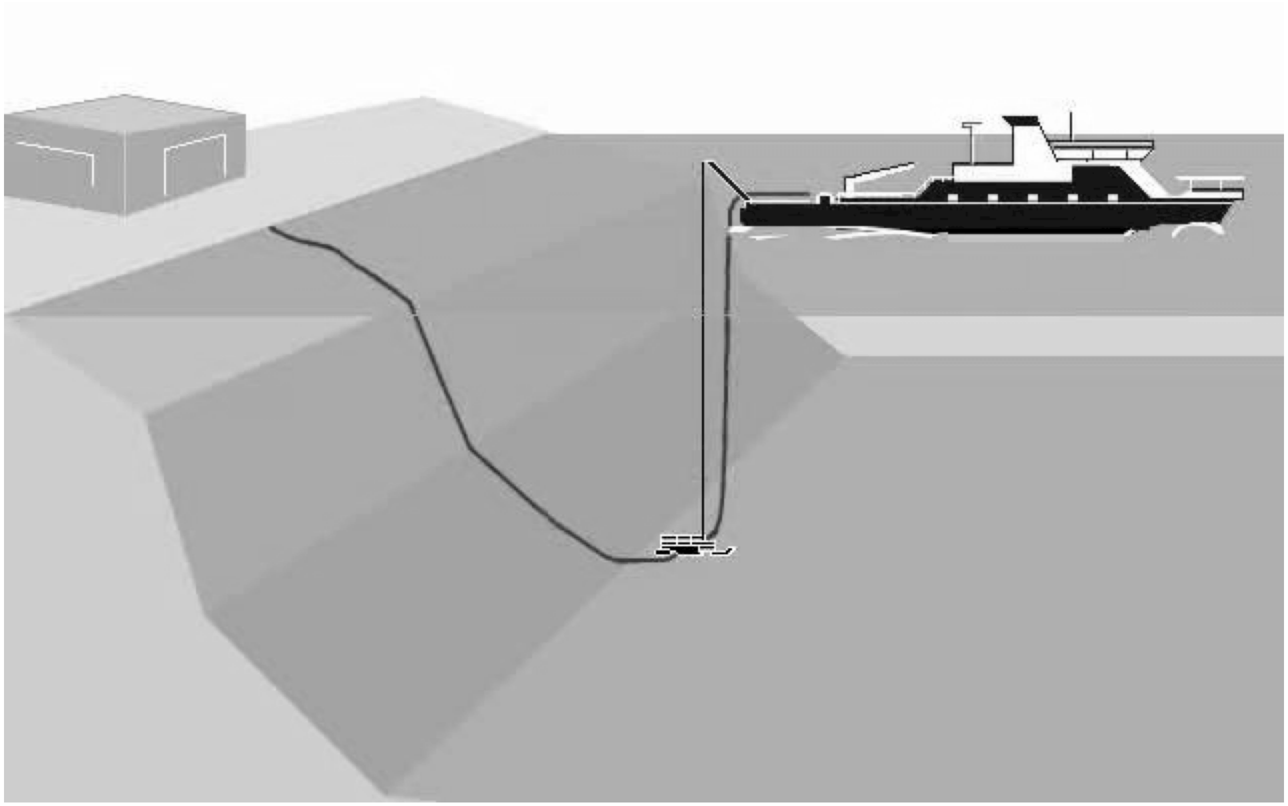
Today's Culture Moment

Laying Transcontinental Cables



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Three Ways to Make a Mirror

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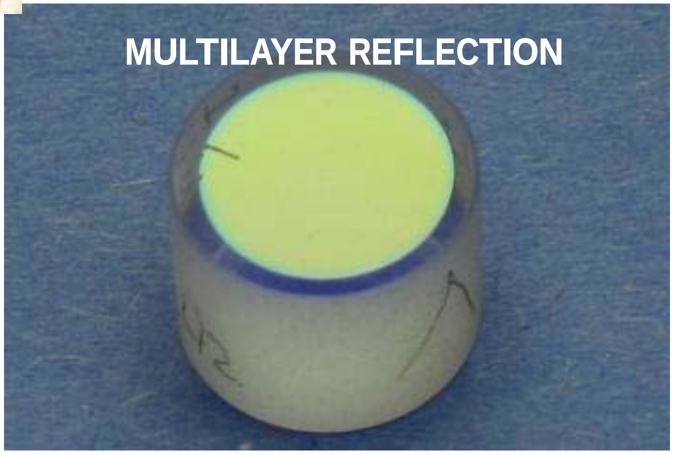

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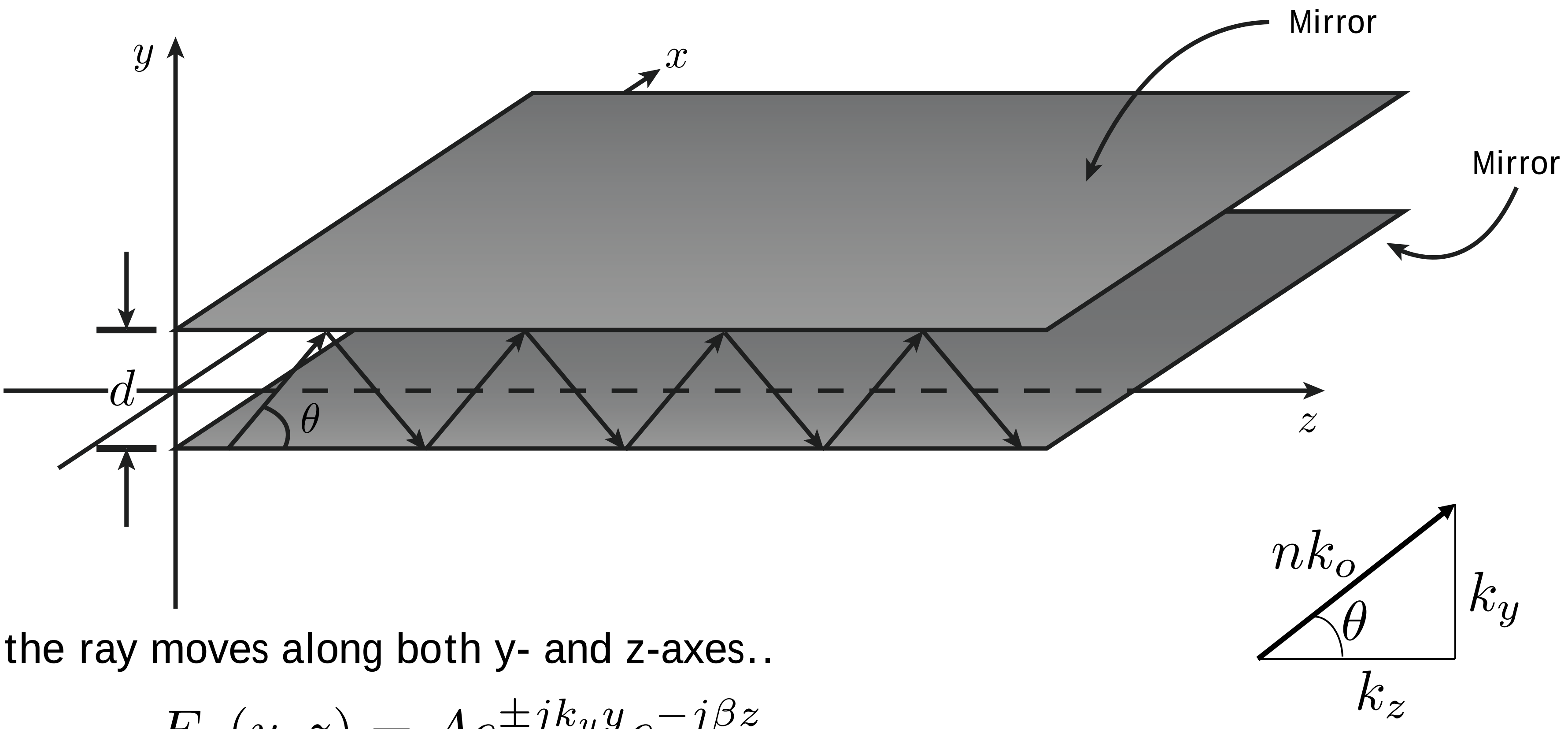
Transporting Light

We can transport light along the z-direction by bouncing it between two mirrors



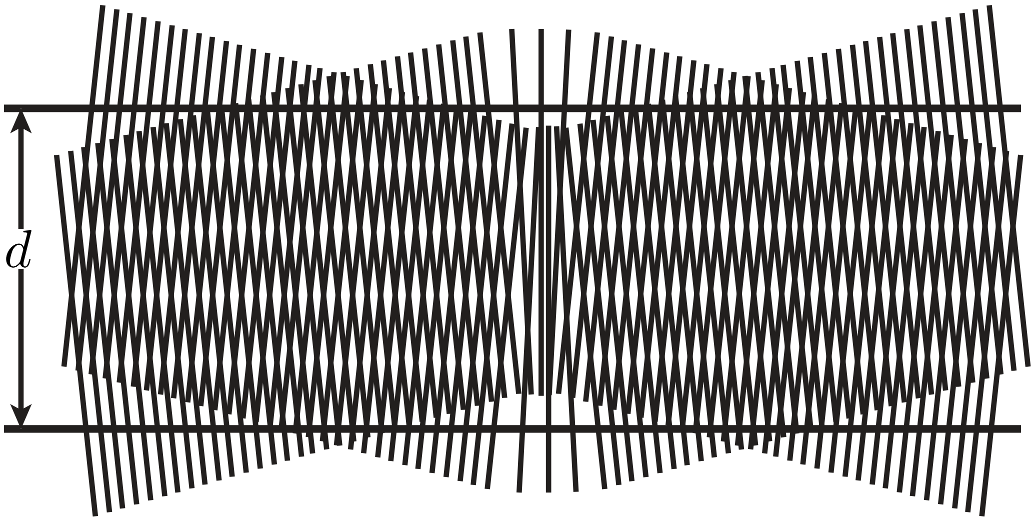
..the ray moves along both y- and z-axes..

$$E_x(y,z) = Ae^{\pm jk_y y}e^{-j\beta z}$$

...where

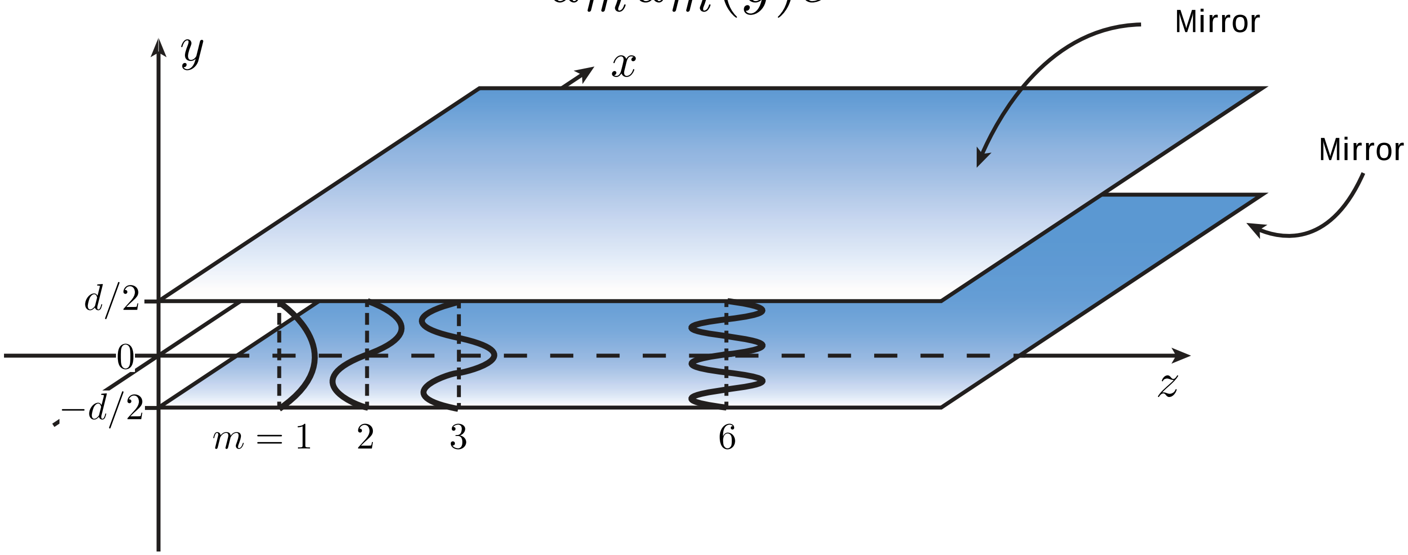
$$k_y = nk_o \sin\theta \qquad k_z = nk_o \cos\theta$$

Transverse Electric (TE) Modes



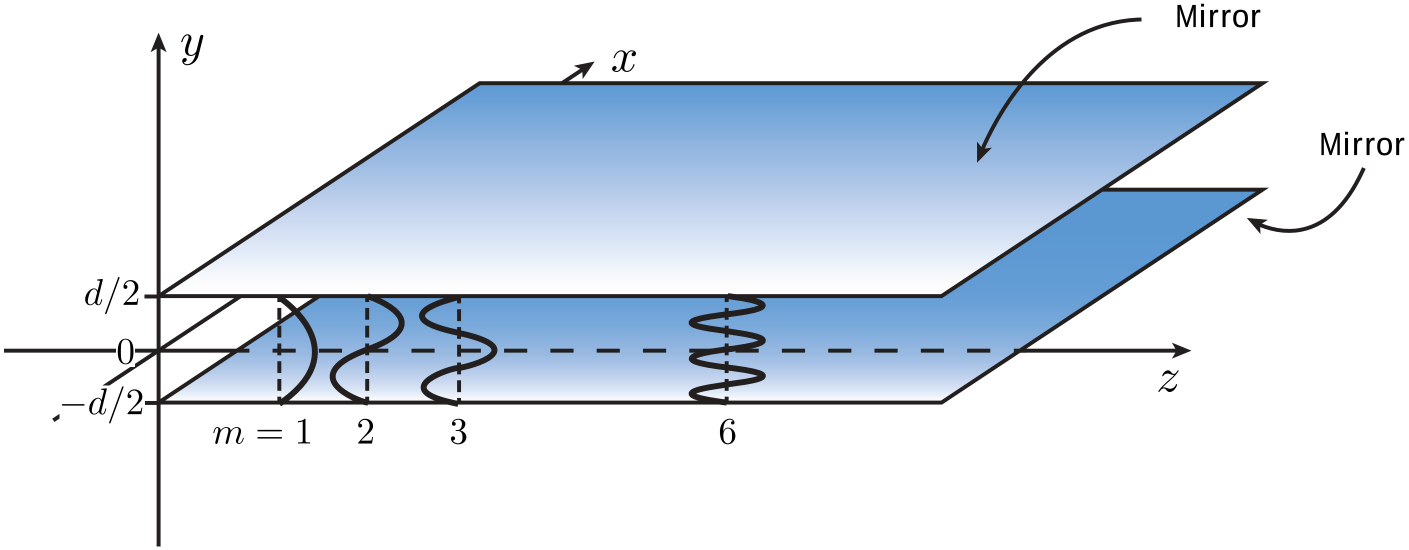
$$E_x(y,z) = \left(A_1 e^{+jk_y y} + A_2 e^{-jk_y y}\right) e^{-j\beta z}$$

$$= a_m u_m(y) e^{-j\beta_m z}$$



STANDING WAVE
IN y-DIRECTION

Perfect Conductor Waveguide



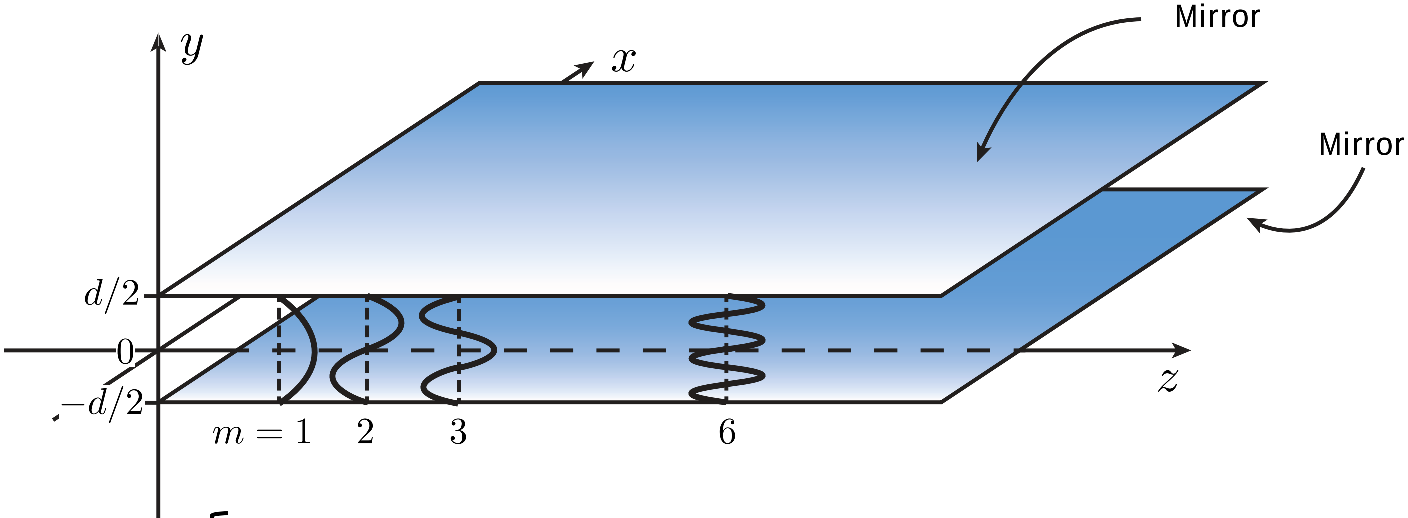
$$u_m(y) = \begin{cases} \sqrt{2/d}\cos k_y y & \text{if } m = \text{odd} \\ \sqrt{2/d}\sin k_y y & \text{if } m = \text{even} \end{cases}$$

$$k_{ym} = m\frac{\pi}{d}$$

Boundary Conditions

$$u_m(y) = \begin{cases} \sqrt{2/d}\cos\left(\frac{m\pi y}{d}\right) & \text{if } m = \text{odd} \\ \sqrt{2/d}\sin\left(\frac{m\pi y}{d}\right) & \text{if } m = \text{even} \end{cases}$$

Transporting Light



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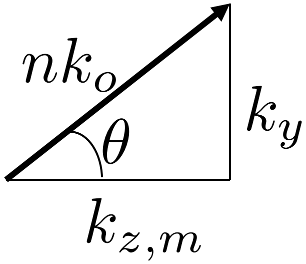
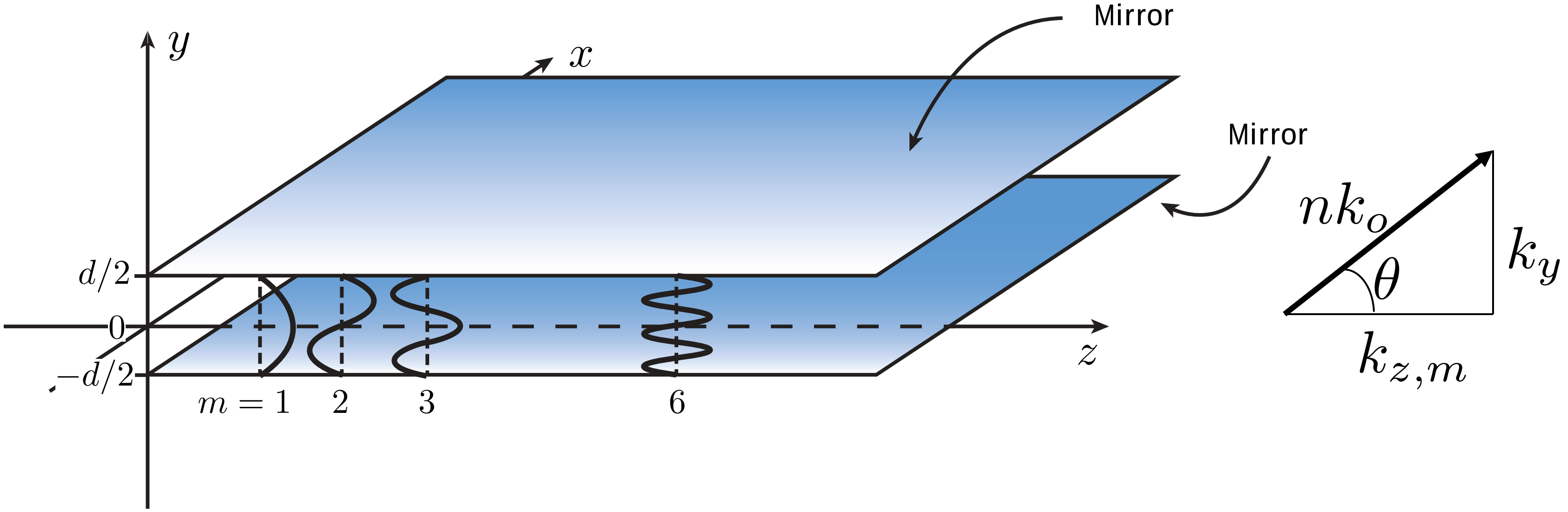


$$k_{ym} = nk_o \sin\theta_m = m\frac{\pi}{d}$$

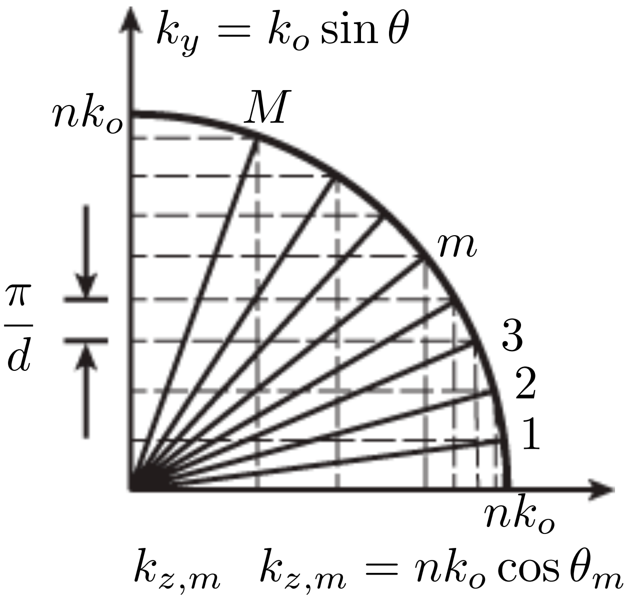
$$k_{z,m} = nk_o \cos\theta_m$$

$$k_{z,m}^2 = k^2 - m^2\frac{\pi^2}{d^2}$$

Transporting Light



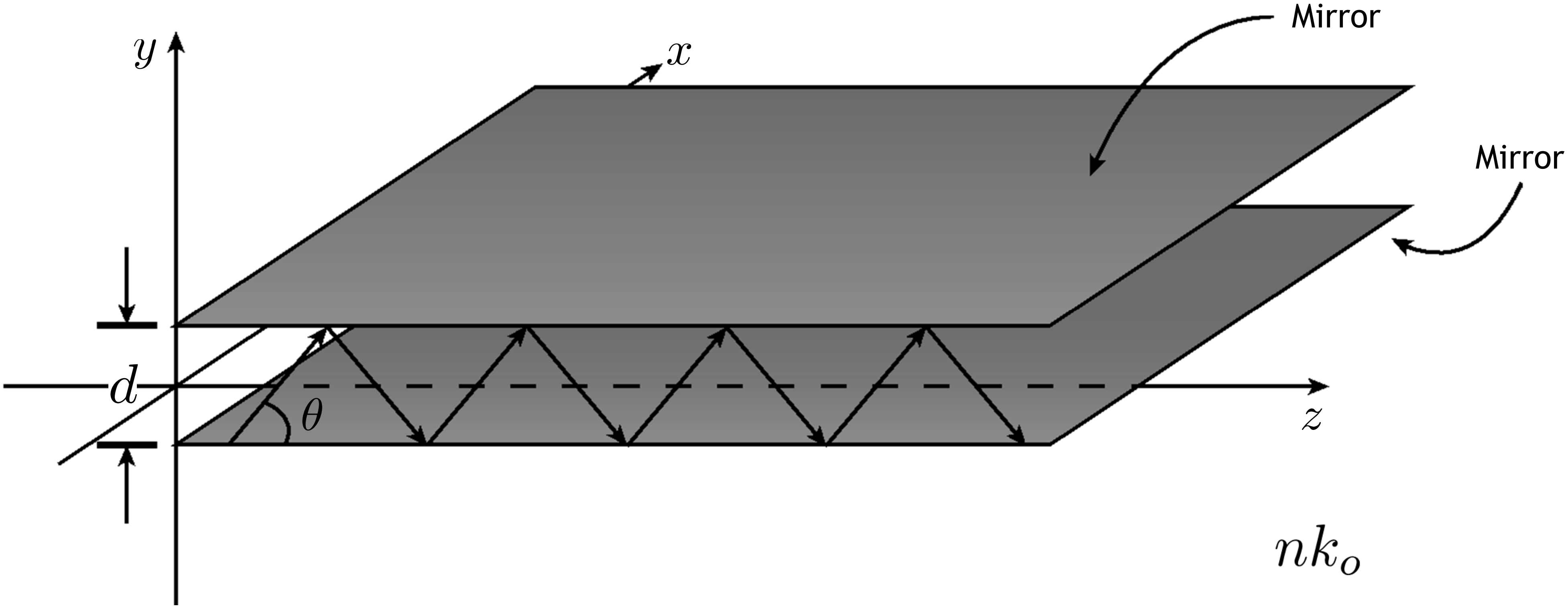
The solutions can be plotted along a circle of radius *k*=*nk*$_o$...



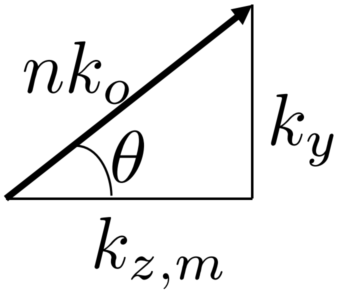
$$k_{ym} = nk_o \sin\theta_m = m\frac{\pi}{d}$$

$$k_{z,m} = nk_o \cos\theta_m$$

Waveguide Mode Propagation Velocity



Velocity along the direction of the guide…

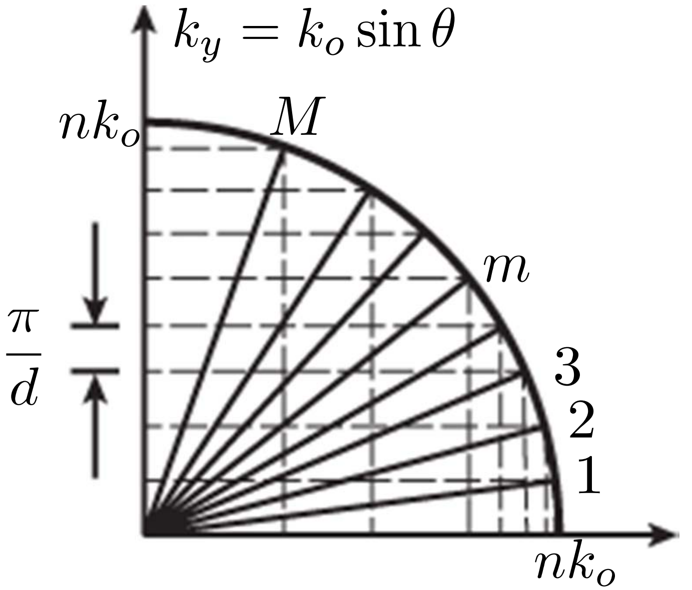


$$v_m = \frac{c}{n}\cos\theta_m$$

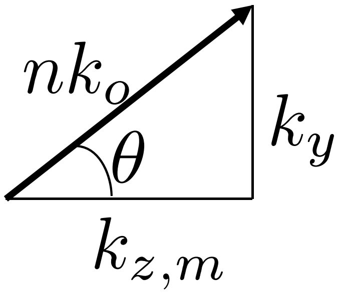
…steeper angles take longer to travel through the guide

Lowest Frequency Guided Mode

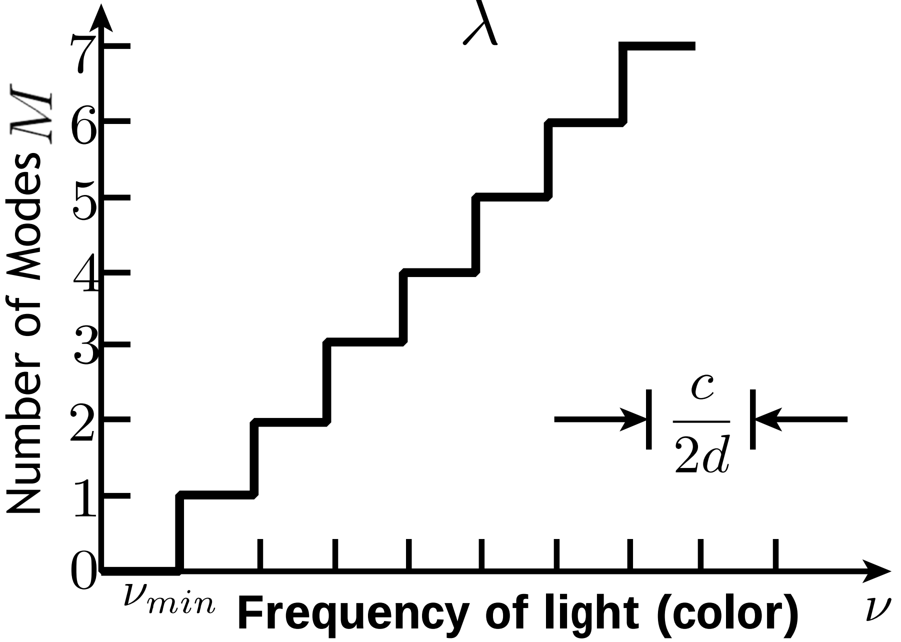
Cut-off Frequency

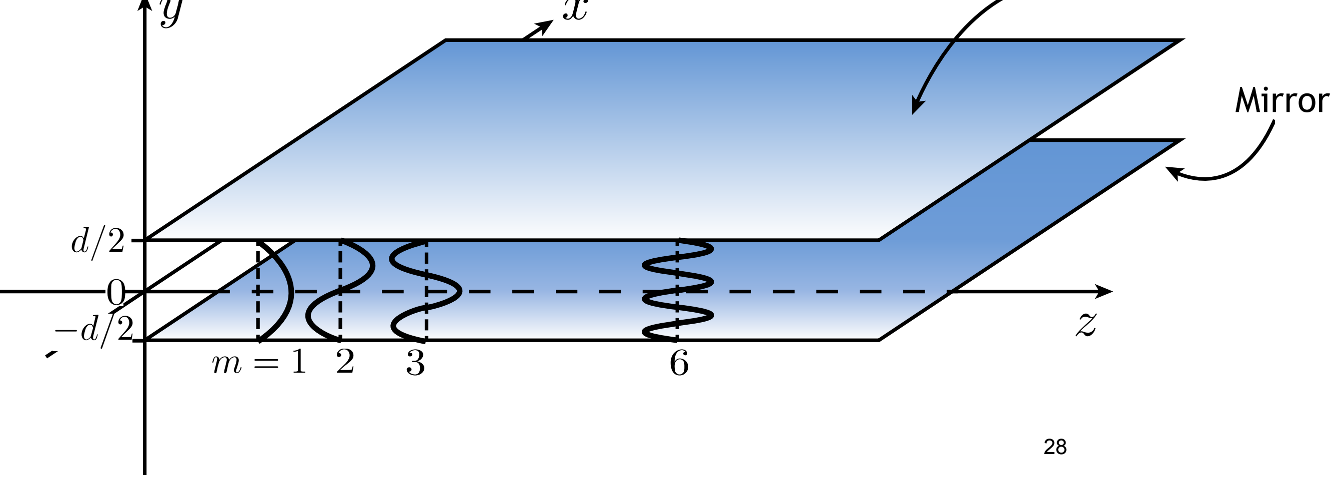


$k_{z,m}$ $\quad k_{z,m} = nk_o \cos\theta_m$



$$M = \frac{2d}{\lambda}$$

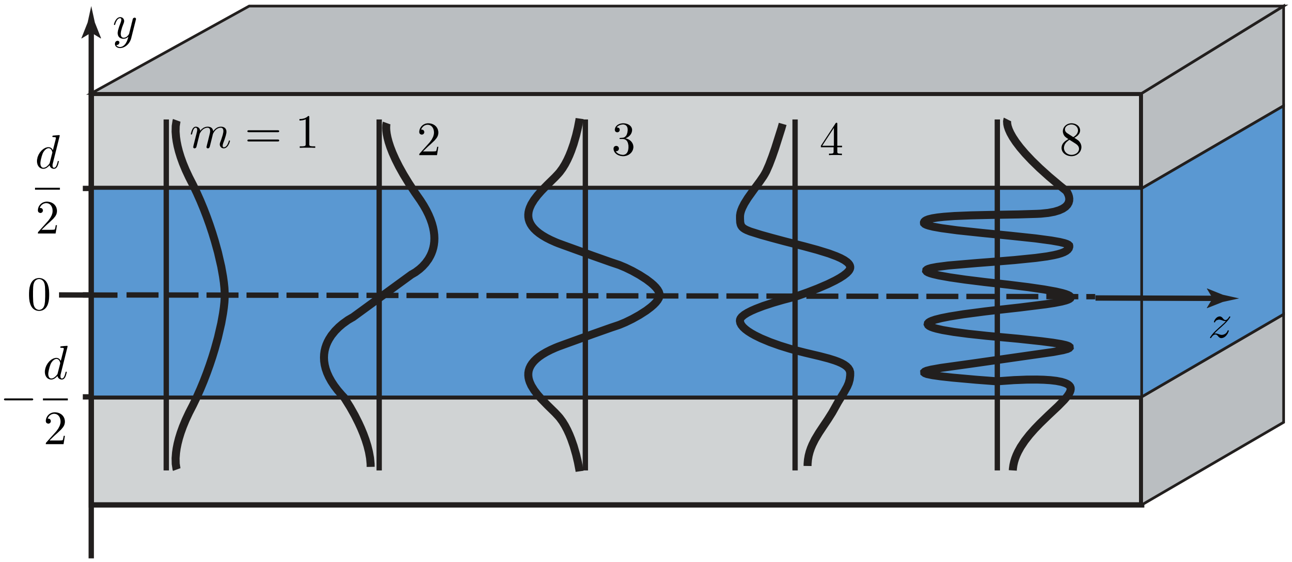




$$\nu_{min} = \frac{c}{2d}$$

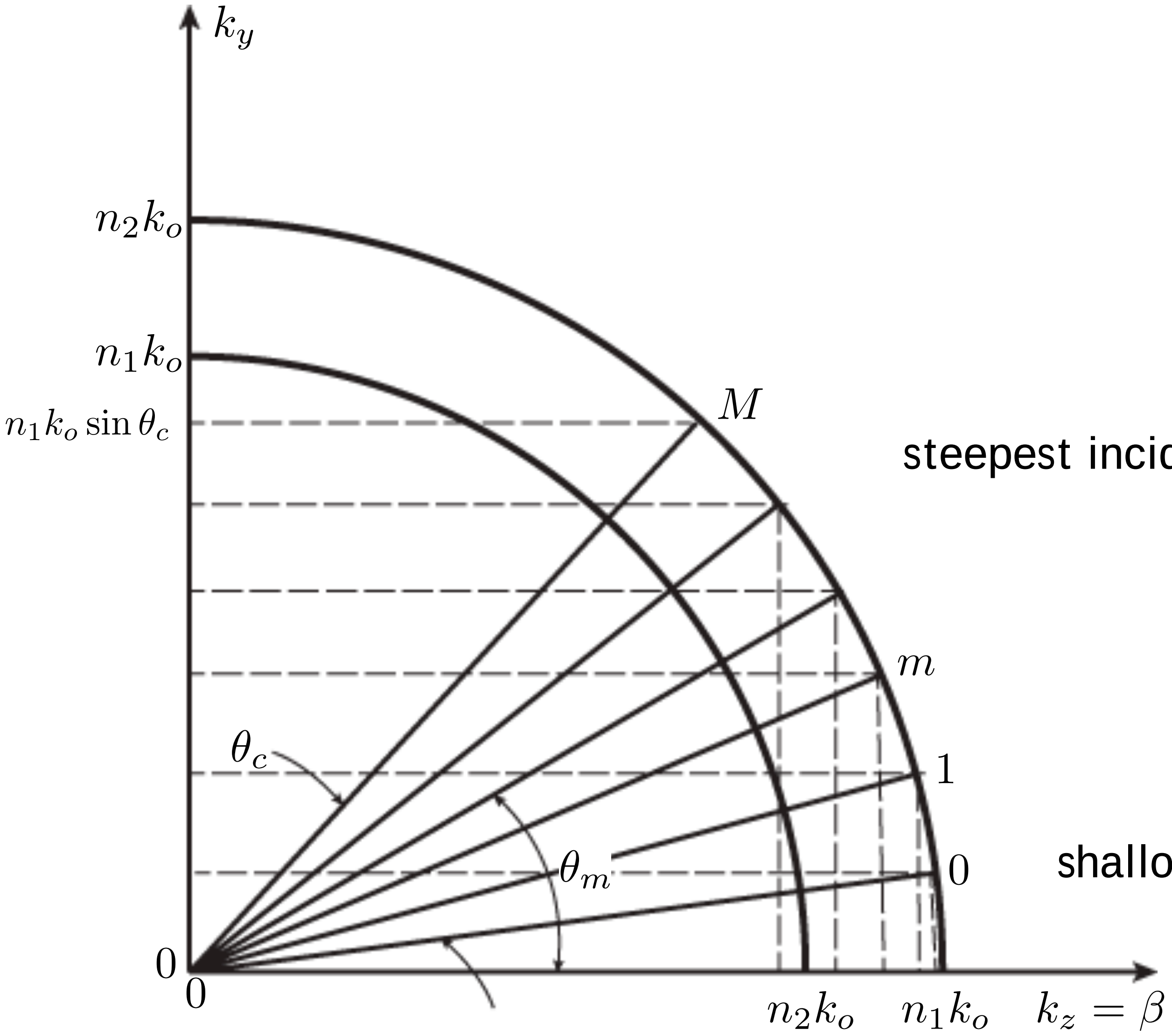
$$\lambda_{max} = 2d$$

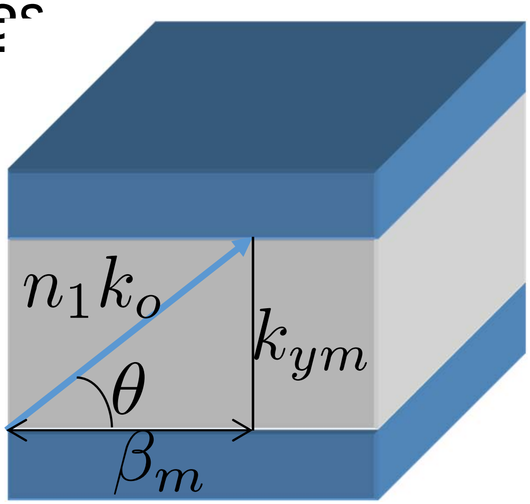
Solutions for a Dielectric Slab Waveguide



What does it mean to be a mode of a waveguide?

Slab Dielectric Waveguides



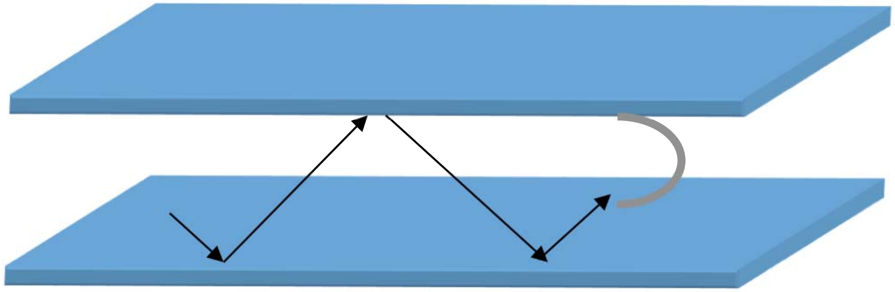


steepest incidence angle

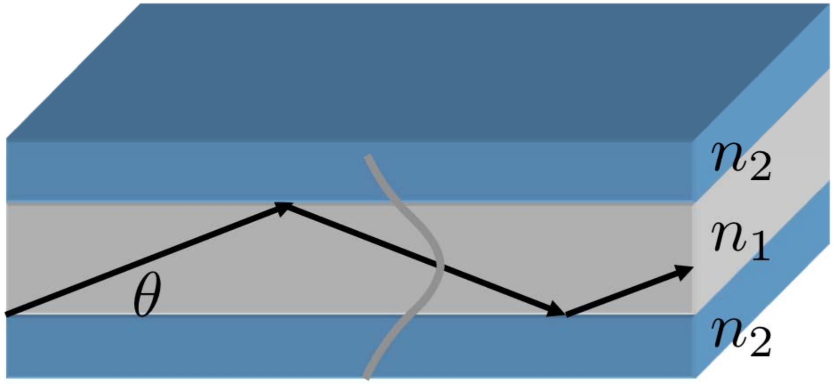
shallowest incidence angle

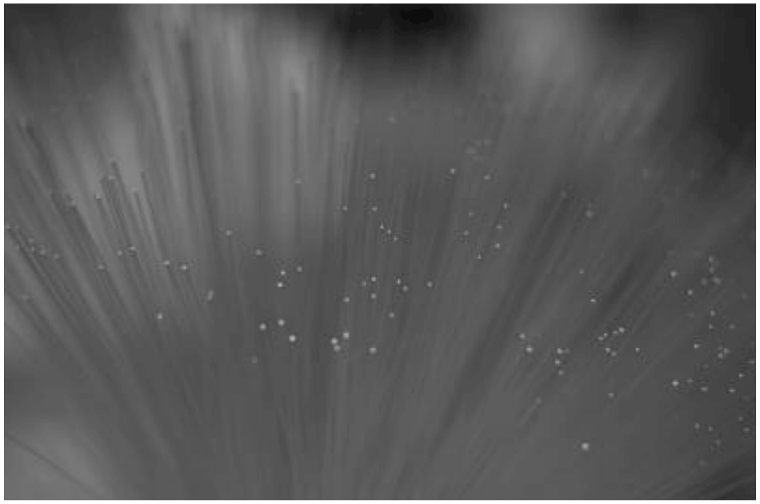
Comparison of Mirror Guide and Dielectric Waveguide

Metal Waveguide

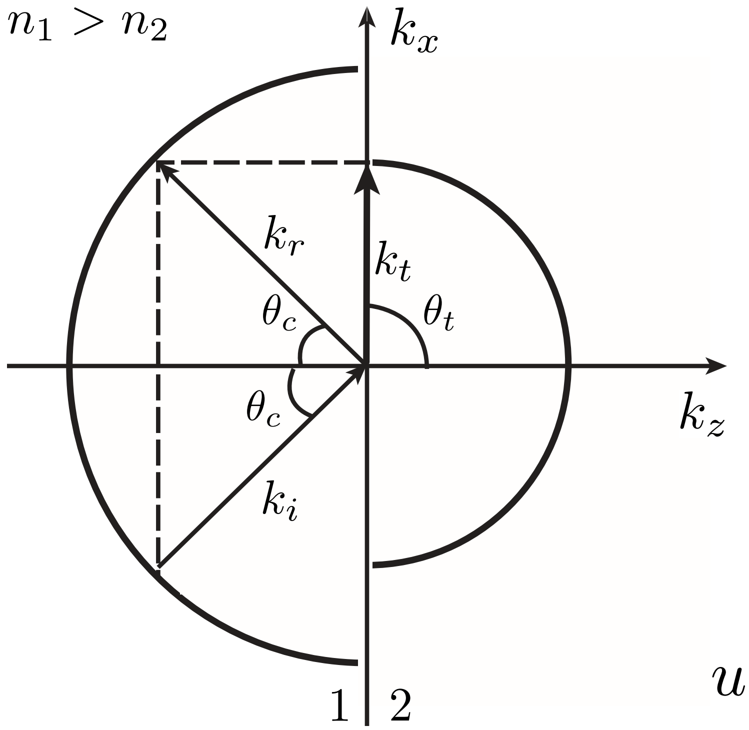


Dielectric Waveguide





Key Takeaways



Total Internal Reflection. What happens for $\theta_i > \theta_C$?

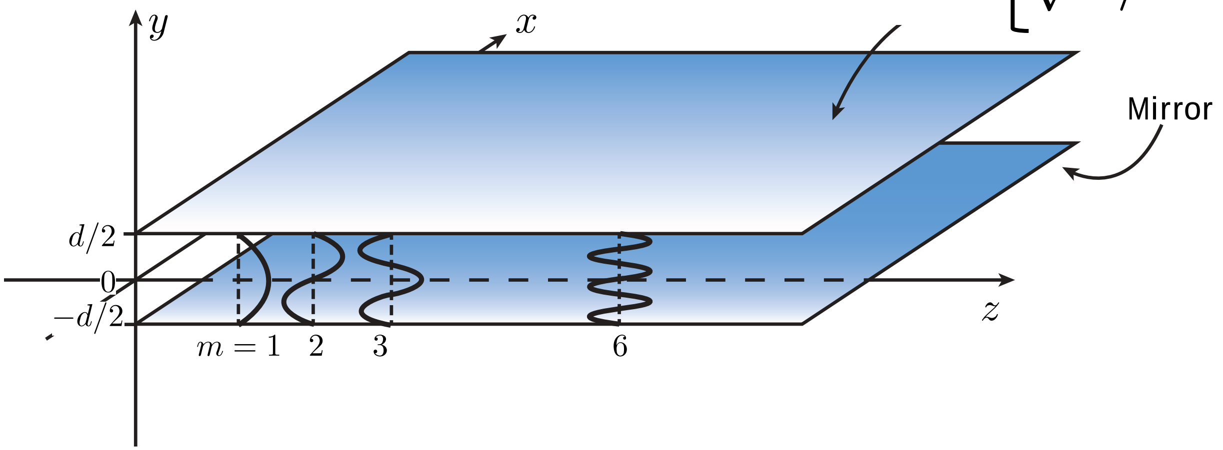
$k_{tz}^2 = k_t^2 - k_{tx}^2 < 0$ → $k_{tz} = \pm j\, \alpha_{tz}$, with α_{tz} real.

Evanescent field

$\exp(-j\, k_{tx}\, x - \alpha_{tz}\, z)$

Waveguide Modes

$$u_m(y) = \begin{cases} \sqrt{2/d}\cos\left(\dfrac{m\pi y}{d}\right) & \text{if } m = \text{odd} \\ \sqrt{2/d}\sin\left(\dfrac{m\pi y}{d}\right) & \text{if } m = \text{even} \end{cases}$$



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