Graph-theoretic Models

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Relevant Reading for Today's Lecture

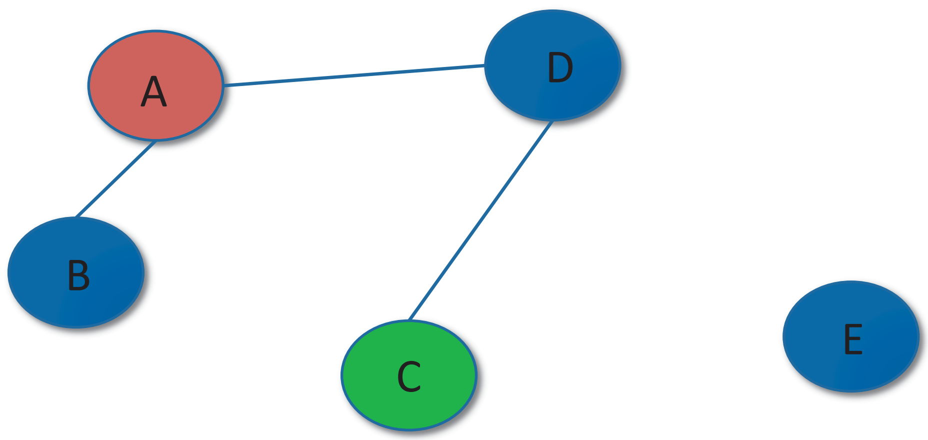
- Section 12.2

Computational Models

- Programs that help us understand the world and solve practical problems
- Saw how we could map the informal problem of choosing what to eat into an optimization problem, and how we could design a program to solve it
- Now want to look at class of models called graphs

What's a Graph?

- Set of nodes (vertices)
  - Might have properties associated with them
- Set of edges (arcs) each consisting of a pair of nodes
  - Undirected (graph)
  - Directed (digraph)
    - Source (parent) and destination (child) nodes
  - Unweighted or weighted



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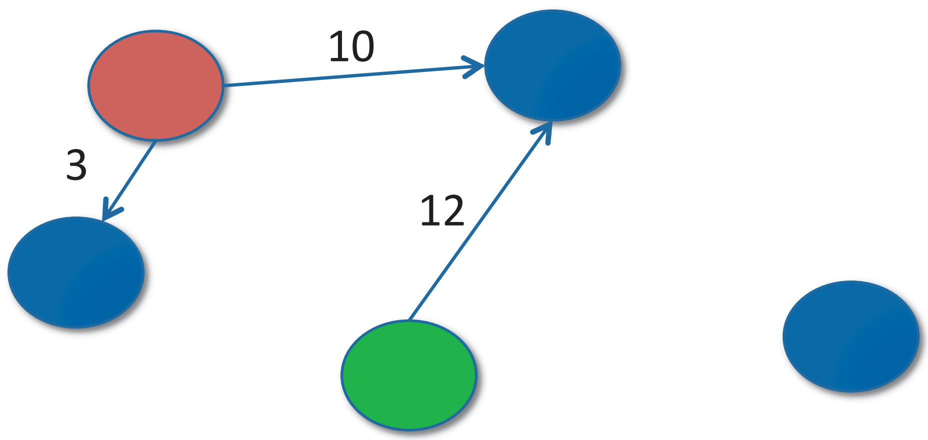
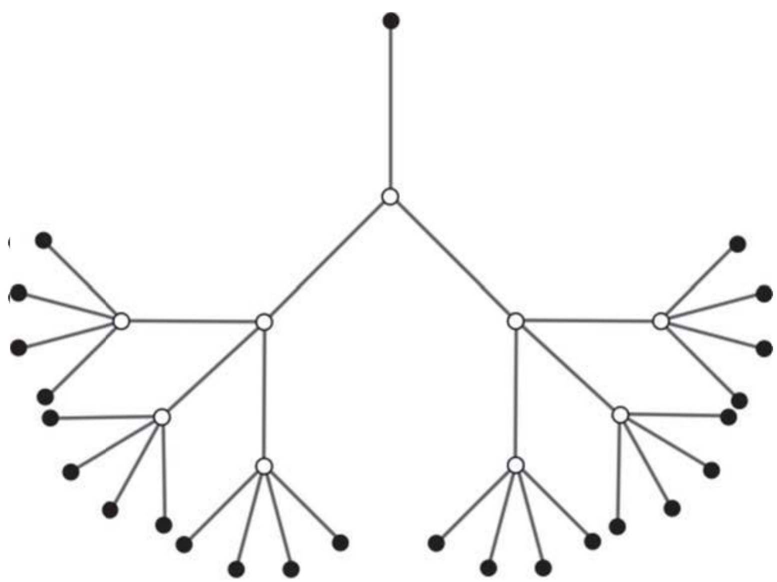


Why Graphs?

- To capture useful relationships among entities
  - Rail links between Paris and London
  - How the atoms in a molecule are related to one another
  - Ancestral relationships

Trees: An Important Special Case

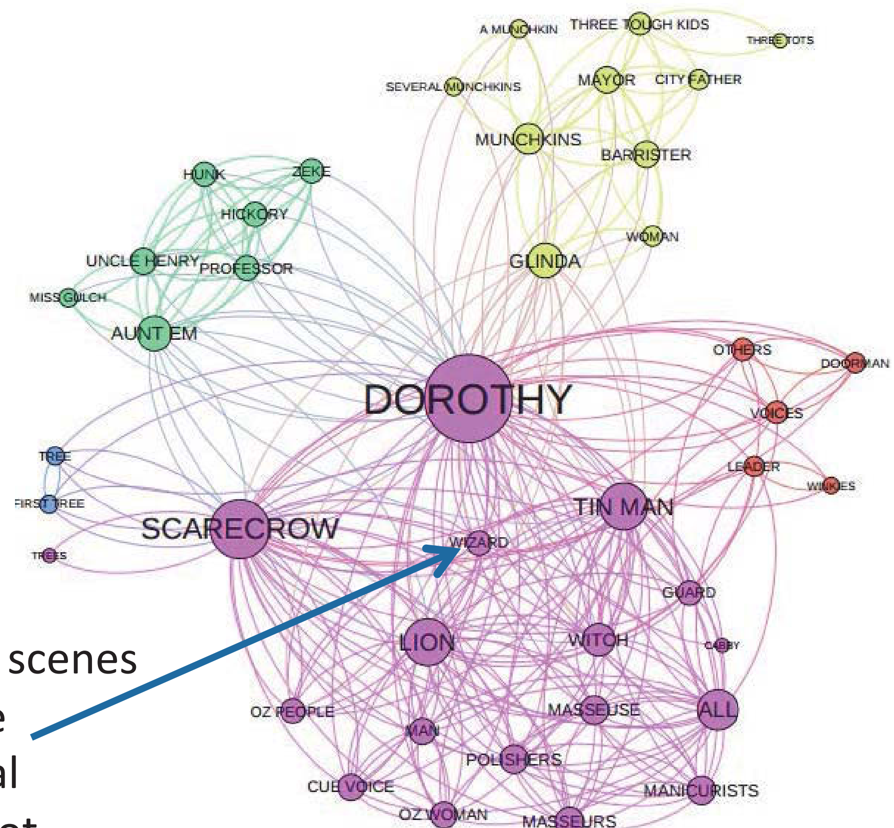
- A special kind of directed graph in which any pair of nodes is connected by a single path
  - Recall the search trees we used to solve knapsack problem



Why Graphs Are So Useful

- World is full of networks based on relationships
  - Computer networks
  - Transportation networks
  - Financial networks
  - Sewer or water networks
  - Political networks
  - Criminal networks
  - Social networks
  - Etc.

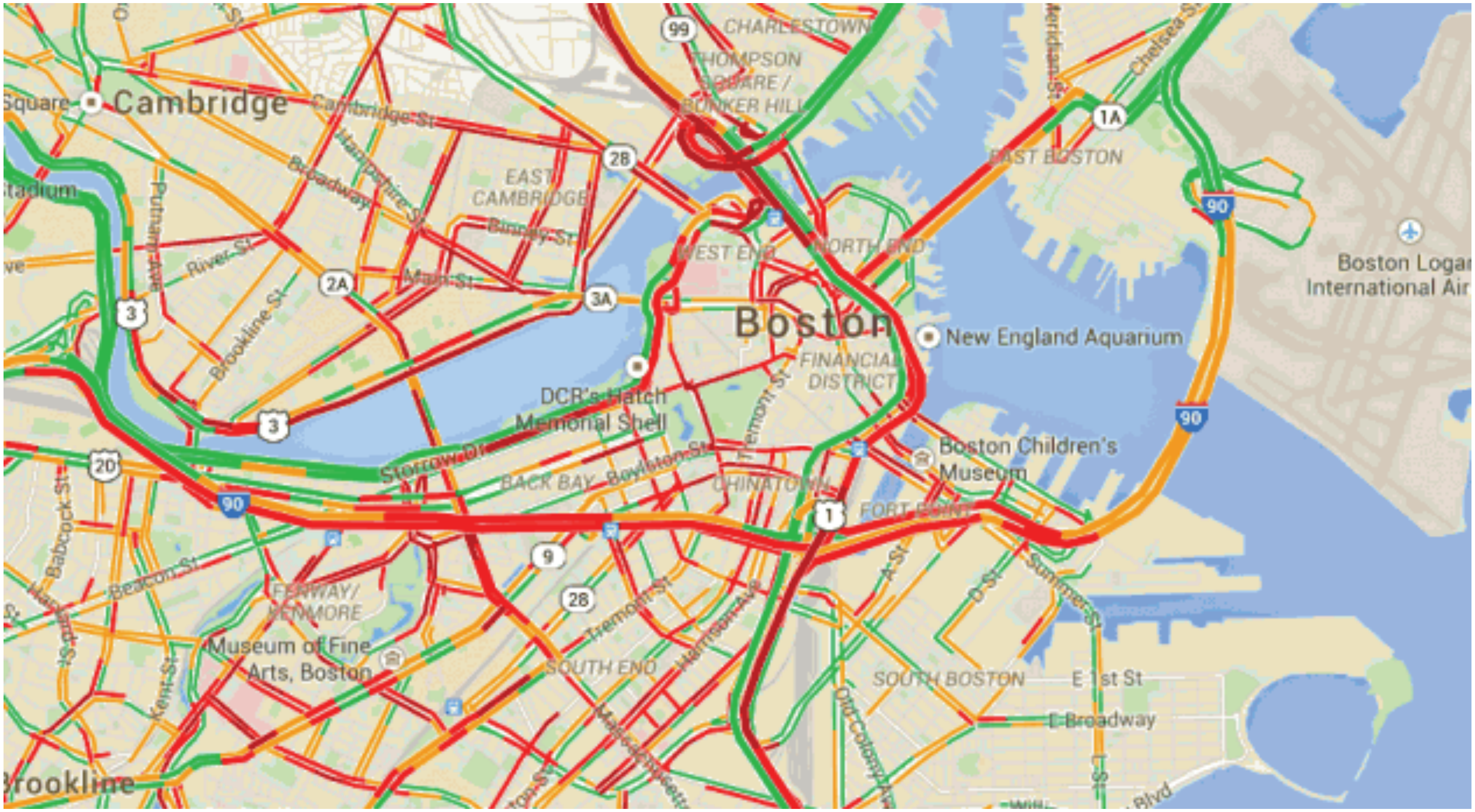
Analysis of "Wizard of Oz":
- size of node reflects number of scenes in which character shares dialogue
- color of clusters reflects natural interactions with each other but not others



Why Graphs Are So Useful

- We will see that not only do graphs capture relationships in connected networks of elements, they also support inference on those structures
  - Finding sequences of links between elements – is there a path from A to B
  - Finding the least expensive path between elements (aka shortest path problem)
  - Partitioning the graph into sets of connected elements (aka graph partition problem)
  - Finding the most efficient way to separate sets of connected elements (aka the min-cut/max-flow problem)

Graph Theory Saves Me Time Every Day

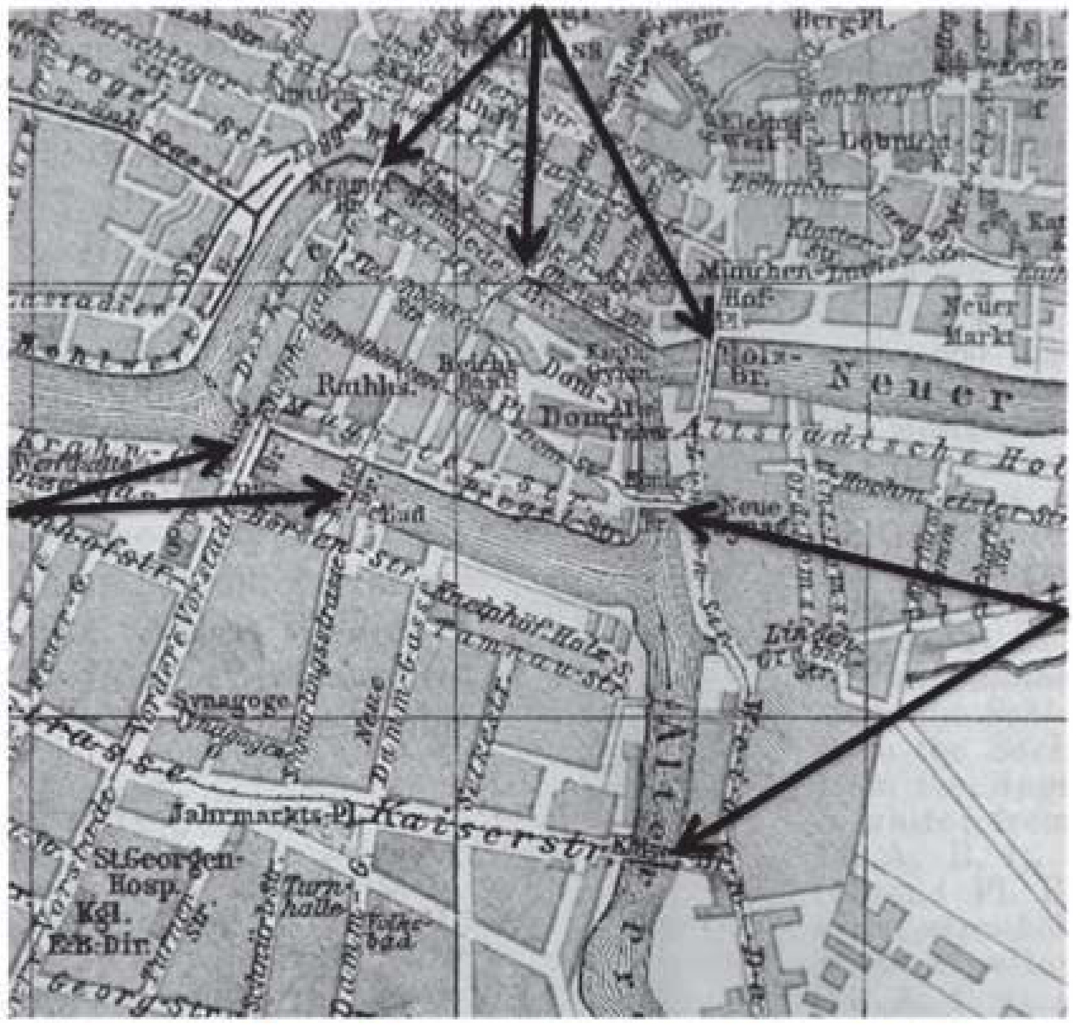


Getting Eric to his Office

- Model road system using a digraph
  - Nodes: points where roads end or meet
  - Edges: connections between points
    - Each edge has a weight
      - Expected time to get from source node to destination node for that edge
      - Distance between source and destination nodes
      - Average speed of travel between source and destination nodes
- Solve a graph optimization problem
  - Shortest weighted path between my house and my office

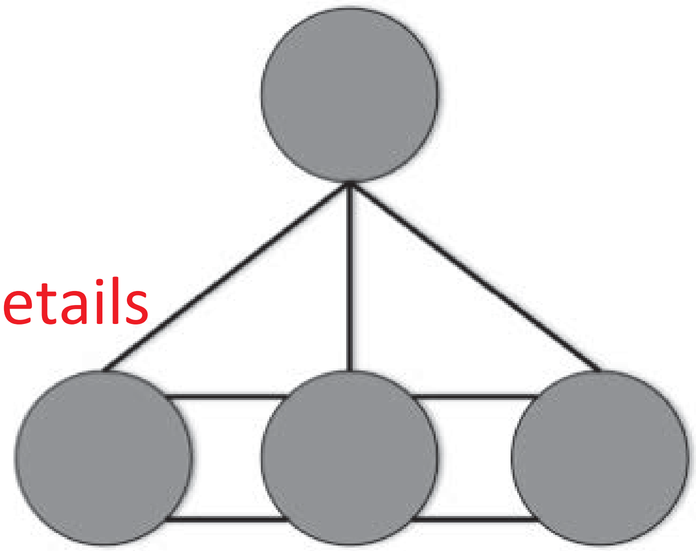
First Reported Use of Graph Theory

- Bridges of Königsberg (1735)
- Possible to take a walk that traverses each of the 7 bridges exactly once?



Leonhard Euler's Model

- Each island a node
- Each bridge an undirected edge
- Model abstracts away irrelevant details
  - Size of islands
  - Length of bridges



- Is there a path that contains each edge exactly once?
  - **No!**

Implementing and using graphs

- Building graphs
  - Nodes
  - Edges
  - Stitching together to make graphs
- Using graphs
  - Searching for paths between nodes
  - Searching for optimal paths between nodes

Class Node

```
class Node(object):
    def __init__(self, name):
        """Assumes name is a string"""
        self.name = name
    def getName(self):
        return self.name
    def __str__(self):
        return self.name
```

Class Edge

```
class Edge(object):
    def __init__(self, src, dest):
        """Assumes src and dest are nodes"""
        self.src = src
        self.dest = dest
    def getSource(self):
        return self.src
    def getDestination(self):
        return self.dest
    def __str__(self):
        return self.src.getName() + '->'\
               + self.dest.getName()
```

Common Representations of Digraphs

- Digraph is a directed graph
  - Edges pass in one direction only
- Adjacency matrix
  - Rows: source nodes
  - Columns: destination nodes
  - Cell[s, d] = 1 if there is an edge from s to d

    = 0 otherwise
  - Note that in digraph, matrix is **not** symmetric
- Adjacency list
  - Associate with each node a list of destination nodes

Class Digraph, part 1

```
class Digraph(object):
    """edges is a dict mapping each node to a list of
    its children"""

    def __init__(self):
        self.edges = {}

    def addNode(self, node):
        if node in self.edges:
            raise ValueError('Duplicate node')
        else:
            self.edges[node] = []

    def addEdge(self, edge):
        src = edge.getSource()
        dest = edge.getDestination()
        if not (src in self.edges and dest in self.edges):
            raise ValueError('Node not in graph')
        self.edges[src].append(dest)
```

Nodes are represented as keys in dictionary

Edges are represented by destinations as values in list associated with a source key

Class Digraph, part 2

```
def childrenOf(self, node):
    return self.edges[node]

def hasNode(self, node):
    return node in self.edges

def getNode(self, name):
    for n in self.edges:
        if n.getName() == name:
            return n
    raise NameError(name)

def __str__(self):
    result = ''
    for src in self.edges:
        for dest in self.edges[src]:
            result = result + src.getName() + '->'\
                     + dest.getName() + '\n'
    return result[:-1] #omit final newline
```

Class Graph

```
class Graph(Digraph):
   def addEdge(self, edge):
       Digraph.addEdge(self, edge)
       rev = Edge(edge.getDestination(), edge.getSource())
       Digraph.addEdge(self, rev)
```

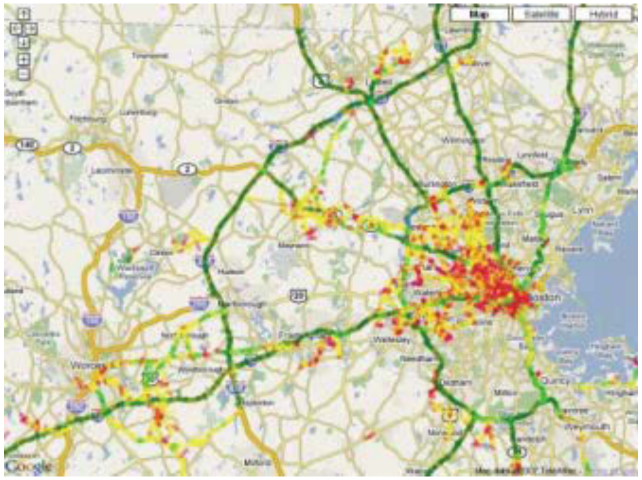
- Graph does not have directionality associated with an edge
  - Edges allow passage in either direction
- Why is Graph a subclass of Digraph?
- Remember the substitution rule?
  - If client code works correctly using an instance of the supertype, it should also work correctly when an instance of the subtype is substituted for the instance of the supertype
- Any program that works with a Digraph will also work with a Graph (but not *vice versa*)

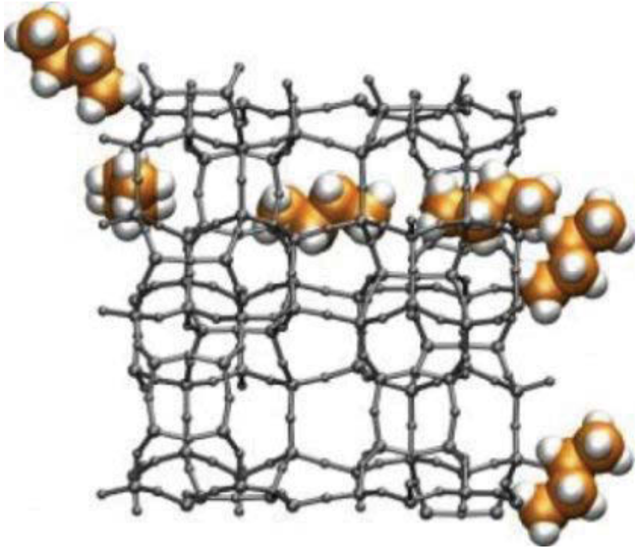
A Classic Graph Optimization Problem

- Shortest path from n1 to n2
  - Shortest sequence of edges such that
    - Source node of first edge is n1
    - Destination of last edge is n2
    - For edges, e1 and e2, in the sequence, if e2 follows e1 in the sequence, the source of e2 is the destination of e1
- Shortest weighted path
  - Minimize the sum of the weights of the edges in the path

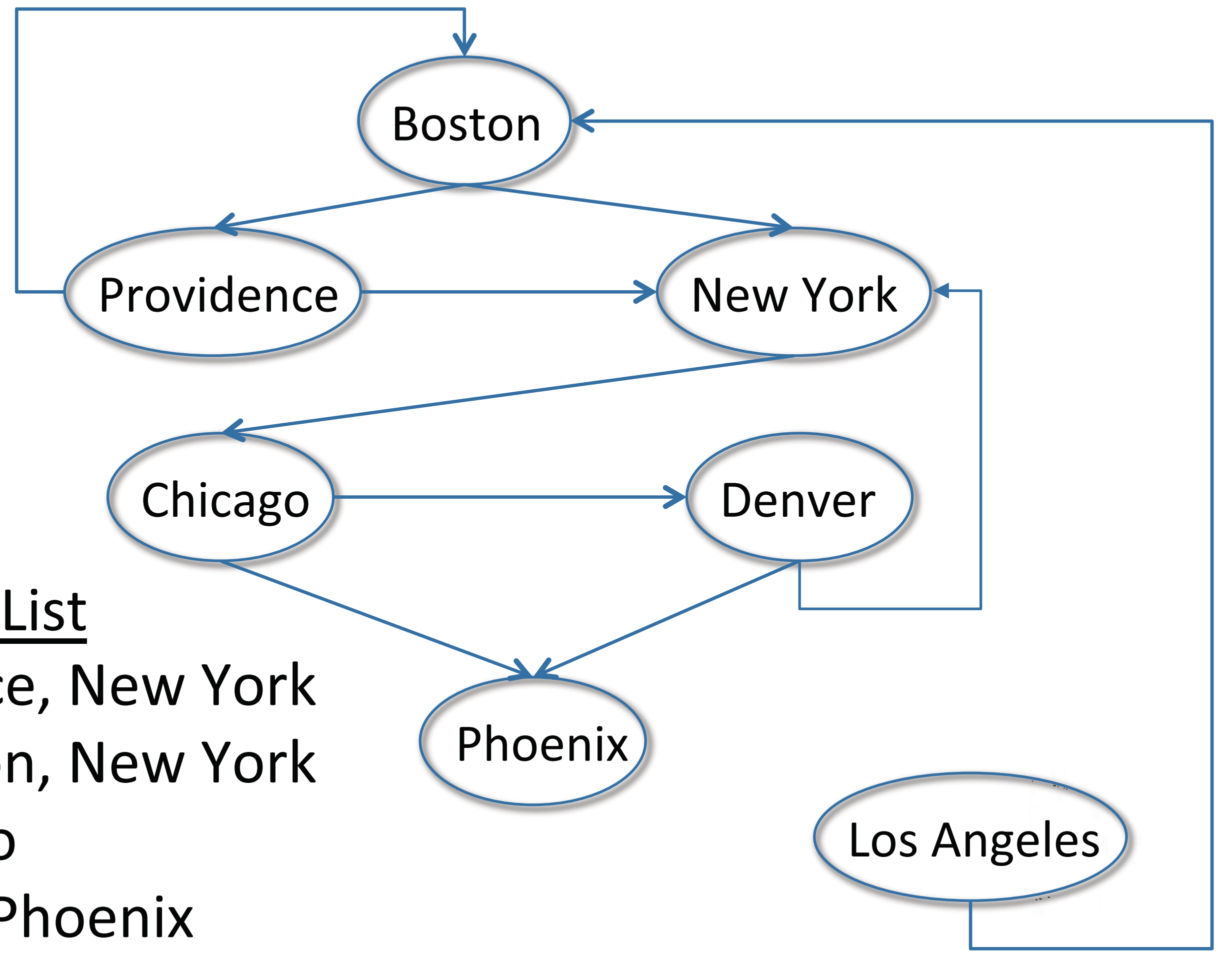
Some Shortest Path Problems

- Finding a route from one city to another
- Designing communication networks
- Finding a path for a molecule through a chemical labyrinth
- ...





An Example



Adjacency List

Boston: Providence, New York

Providence: Boston, New York

New York: Chicago

Chicago: Denver, Phoenix

Denver: Phoenix, New York

Los Angeles: Boston

Phoenix:

Build the Graph

```
def buildCityGraph(graphType):
    g = graphType()
    for name in ('Boston', 'Providence', 'New York', 'Chicago',
                 'Denver', 'Phoenix', 'Los Angeles'): #Create 7 nodes
        g.addNode(Node(name))
    g.addEdge(Edge(g.getNode('Boston'), g.getNode('Providence')))
    g.addEdge(Edge(g.getNode('Boston'), g.getNode('New York')))
    g.addEdge(Edge(g.getNode('Providence'), g.getNode('Boston')))
    g.addEdge(Edge(g.getNode('Providence'), g.getNode('New York')))
    g.addEdge(Edge(g.getNode('New York'), g.getNode('Chicago')))
    g.addEdge(Edge(g.getNode('Chicago'), g.getNode('Denver')))
    g.addEdge(Edge(g.getNode('Chicago'), g.getNode('Phoenix')))
    g.addEdge(Edge(g.getNode('Denver'), g.getNode('Phoenix')))
    g.addEdge(Edge(g.getNode('Denver'), g.getNode('New York')))
    g.addEdge(Edge(g.getNode('Los Angeles'), g.getNode('Boston')))
```

Finding the Shortest Path

- Algorithm 1, depth-first search (DFS)
- Similar to left-first depth-first method of enumerating a search tree (Lecture 2)
- Main difference is that graph might have cycles, so we must keep track of what nodes we have visited to avoid going in infinite loops

Note that we are using divide-and-conquer: if we can find a path from a source to an intermediate node, and a path from the intermediate node to the destination, the combination is a path from source to destination

Depth First Search

- Start at an initial node
- Consider all the edges that leave that node, in some order
- Follow the first edge, and check to see if at goal node
- If not, repeat the process from new node
- Continue until either find goal node, or run out of options
    - When run out of options, backtrack to the previous node and try the next edge, repeating this process

Depth First Search (DFS)

```
def DFS(graph, start, end, path, shortest, toPrint = False):
    path = path + [start]
    if toPrint:
        print('Current DFS path:', printPath(path))
    if start == end:
        return path
    for node in graph.childrenOf(start):
        if node not in path: #avoid cycles
            if shortest == None or len(path) < len(shortest):
                newPath = DFS(graph, node, end, path, shortest, toPrint)
                if newPath != None:
                    shortest = newPath
        elif toPrint:
            print('Already visited', node)
    return shortest

def shortestPath(graph, start, end, toPrint = False):
    return DFS(graph, start, end, [], None, toPrint)
```

Note how will explore all paths through first node, before …

… returning to this point in the recursion to try next node

DFS called from a wrapper function: `shortestPath`

Gets recursion started properly

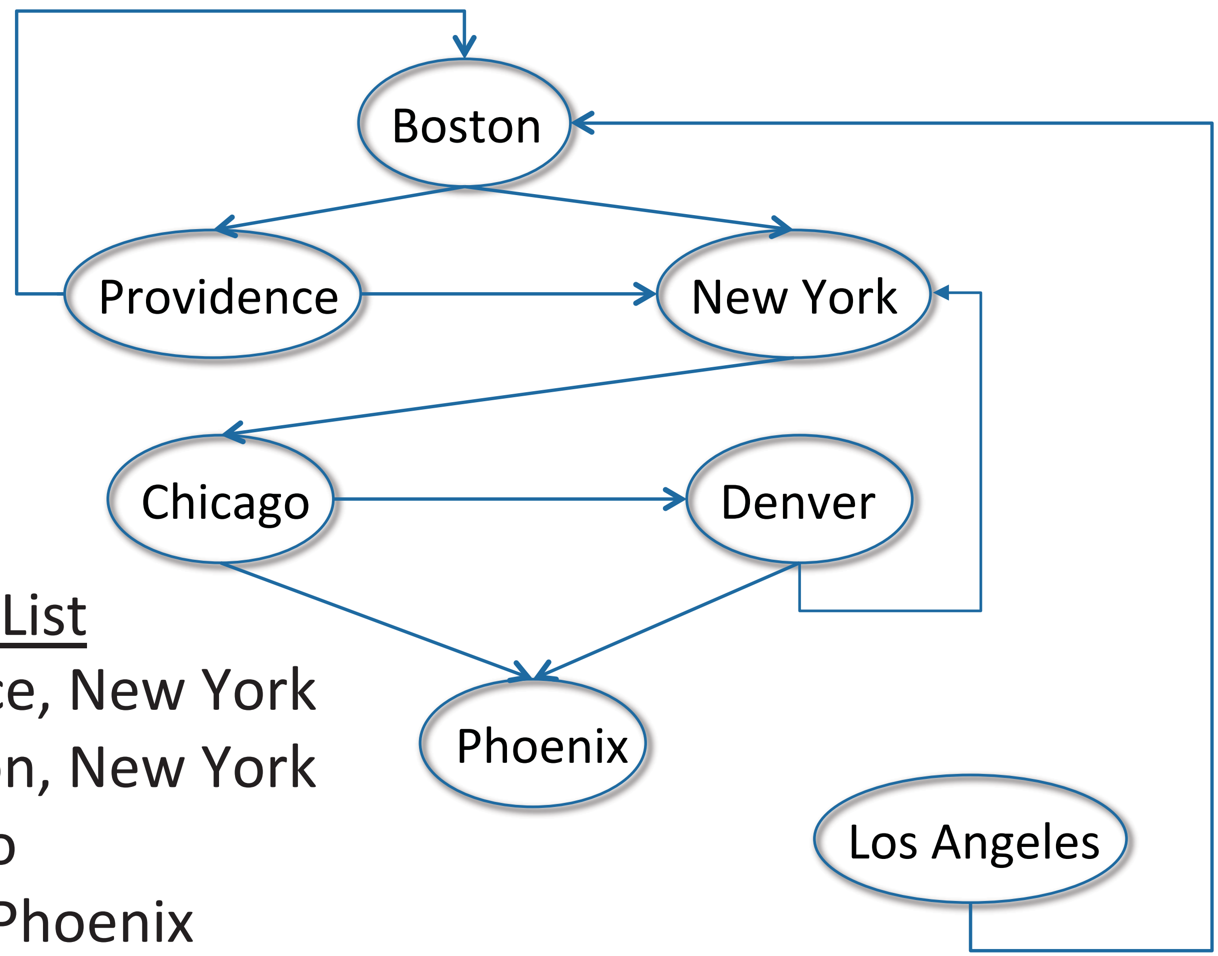
Provides appropriate abstraction

Test DFS

```
def testSP(source, destination):
    g = buildCityGraph(DiGraph)
    sp = shortestPath(g, g.getNode(source), g.getNode(destination)
                      toPrint = True)
    if sp != None:
        print('Shortest path from', source, 'to',
              destination, 'is', printPath(sp))
    else:
        print('There is no path from', source, 'to', destination)

testSP('Boston', ’Chicago')
```

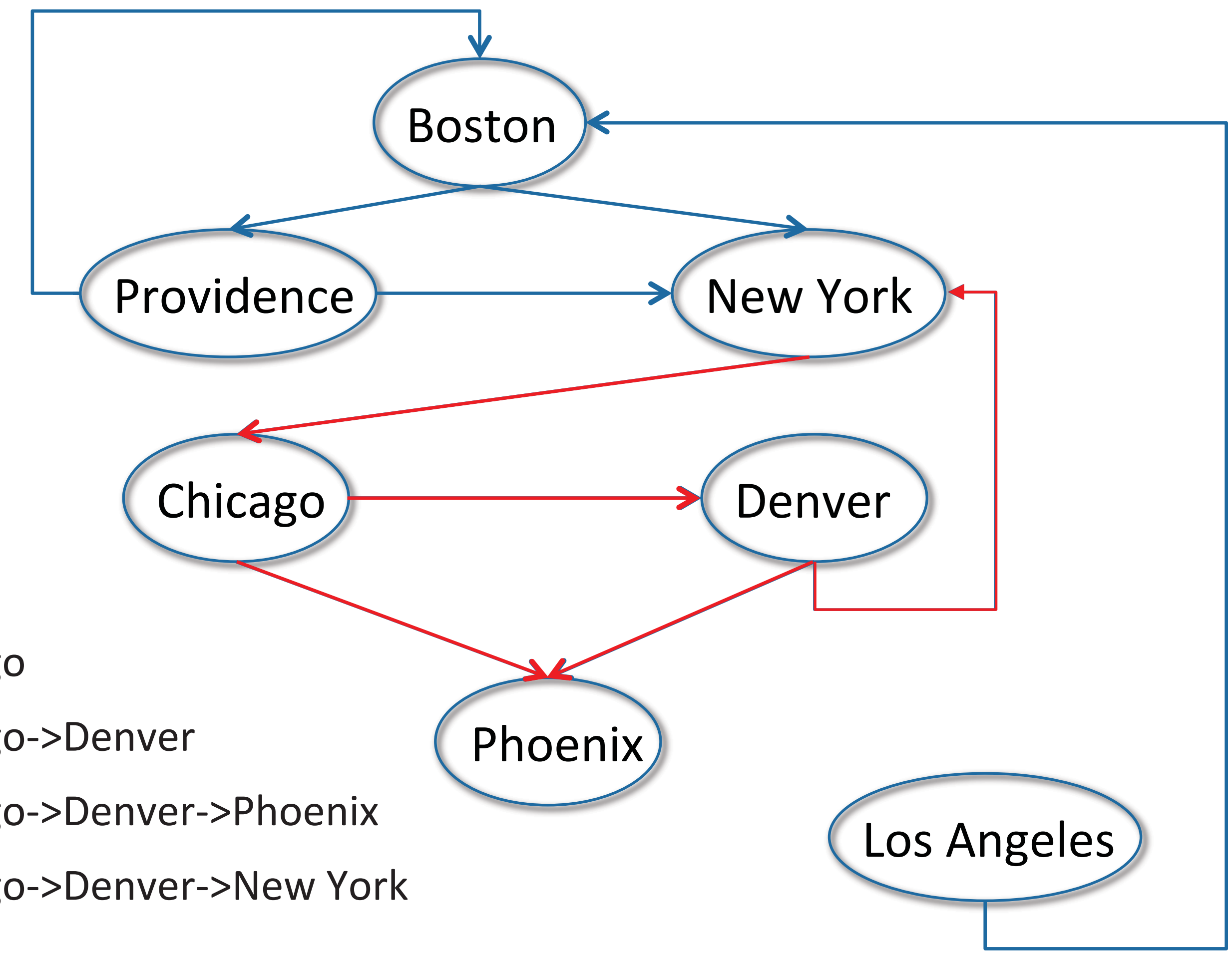
An Example



<u>Adjacency List</u>

Boston: Providence, New York
Providence: Boston, New York
New York: Chicago
Chicago: Denver, Phoenix
Denver: Phoenix, New York
Los Angeles: Boston
Phoenix:

Output (Chicago to Boston)



Current DFS path: Chicago

Current DFS path: Chicago->Denver

Current DFS path: Chicago->Denver->Phoenix

Current DFS path: Chicago->Denver->New York

Already visited Chicago

Current DFS path: Chicago->Phoenix

There is no path from Chicago to Boston

Output (Boston to Phoenix)

Current DFS path: Boston

Current DFS path: Boston->Providence

Already visited Boston

Current DFS path: Boston->Providence->New York

Current DFS path: Boston->Providence->New York->Chicago

Current DFS path: Boston->Providence->New York->Chicago->Denver

Current DFS path: Boston->Providence->New York->Chicago->Denver->Phoenix Found path

Already visited New York

Current DFS path: Boston->Providence->New York->Chicago->Phoenix Found a shorter path

Current DFS path: Boston->New York

Current DFS path: Boston->New York->Chicago

Current DFS path: Boston->New York->Chicago->Denver

Current DFS path: Boston->New York->Chicago->Denver->Phoenix Found a “shorter” path

Already visited New York

Current DFS path: Boston->New York->Chicago->Phoenix Found a shorter path

Shortest path from Boston to Phoenix is Boston->New York->Chicago->Denver->Phoenix

Breadth First Search

- Start at an initial node
- Consider all the edges that leave that node, in some order
- Follow the first edge, and check to see if at goal node
- If not, try the next edge from the current node
- Continue until either find goal node, or run out of options
  - When run out of edge options, move to next node at same distance from start, and repeat
  - When run out of node options, move to next level in the graph (all nodes one step further from start), and repeat

Algorithm 2: Breadth-first Search (BFS)

```
def BFS(graph, start, end, toPrint = False):
    initPath = [start]
    pathQueue = [initPath]
    while len(pathQueue) != 0:
        #Get and remove oldest element in pathQueue
        tmpPath = pathQueue.pop(0)
        if toPrint:
            print('Current BFS path:', printPath(tmpPath))
        lastNode = tmpPath[-1]
        if lastNode == end:
            return tmpPath
        for nextNode in graph.childrenOf(lastNode):
            if nextNode not in tmpPath:
                newPath = tmpPath + [nextNode]
                pathQueue.append(newPath)
    return None
```

?

Explore all paths with n hops before exploring any path with more than n hops

Output (Boston to Phoenix)

Current BFS path: Boston

Current BFS path: Boston->Providence

Current BFS path: Boston->New York

Current BFS path: Boston->Providence->New York

Current BFS path: Boston->New York->Chicago

Current BFS path: Boston->Providence->New York->Chicago

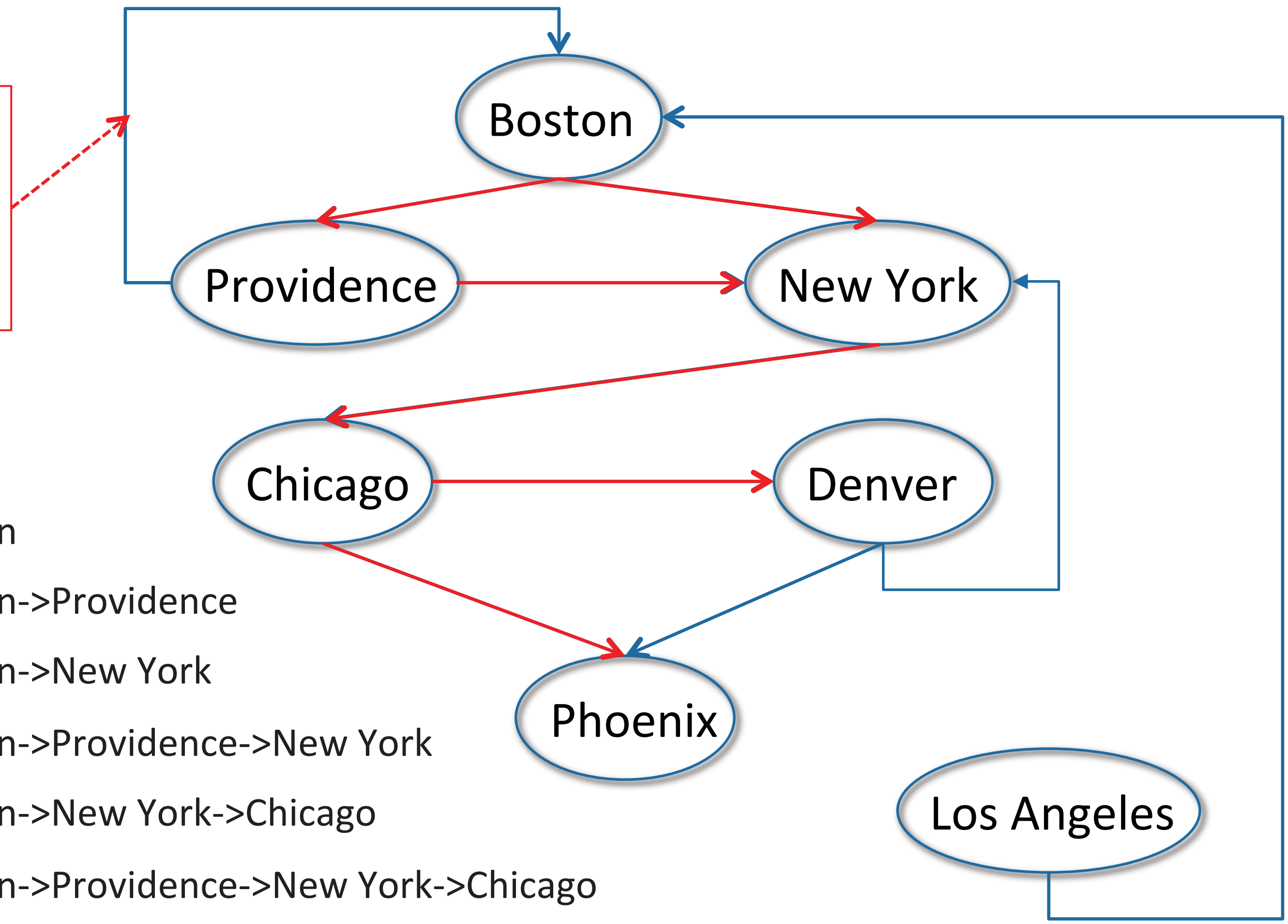
Current BFS path: Boston->New York->Chicago->Denver

Current BFS path: Boston->New York->Chicago->Phoenix

Shortest path from Boston to Phoenix is Boston->New York->Chicago->Phoenix

Output (Boston to Pheonix)

Note that we skip a path that revisits a node



Current BFS path: Boston

Current BFS path: Boston->Providence

Current BFS path: Boston->New York

Current BFS path: Boston->Providence->New York

Current BFS path: Boston->New York->Chicago

Current BFS path: Boston->Providence->New York->Chicago

Current BFS path: Boston->New York->Chicago->Denver

Current BFS path: Boston->New York->Chicago->Phoenix

Shortest path from Boston to Phoenix is Boston->New York->Chicago->Phoenix

What About a Weighted Shortest Path

- Want to minimize the sum of the weights of the edges, not the number of edges
- DFS can be easily modified to do this
- BFS cannot, since shortest weighted path may have more than the minimum number of hops

Recap

- Graphs are cool
  - Best way to create a model of many things
    - Capture relationships among objects
  - Many important problems can be posed as graph optimization problems we already know how to solve
- Depth-first and breadth-first search are important algorithms
  - Can be used to solve many problems