

14.54 Fall 2016
Solution Recitation 1

1. Consider an economy (Home) with 2 goods, Cloth (C) and Food (F). All consumers have same Cobb-Douglas utility function:

$$U(D_C, D_F) = (D_C)^a (D_F)^b$$

with $a, b > 0$. Aggregate endowments of C and F are given by $E_C = 100$ and $E_F = 200$.

(a) Show that:

$$\frac{p_C D_C}{p_C D_C + p_F D_F} = \frac{a}{a+b} \text{ and } \frac{p_F D_F}{p_C D_C + p_F D_F} = \frac{b}{a+b}$$

Solution Each consumer solves the following problem:

$$\begin{aligned} \max_{D_C, D_F} & (D_C)^a (D_F)^b \\ \text{st} & \\ & p_C D_C + p_F D_F = p_C e_C + p_F e_F \end{aligned}$$

where e refers to that individual's endowment.

First of all, we know the solution will be interior because if consumption of either good is 0 then utility is 0 no matter how much of the other good they consume. Intuitively this implies that the marginal utility from either good evaluated at 0 is infinite. More formally we could check that the necessary conditions for a boundary solution are never satisfied for positive prices by noticing that

$$\begin{aligned} MRS(D_C, D_F) &= \frac{a D_C^{a-1} D_F^b}{b D_C^a D_F^{b-1}} = \frac{a D_F}{b D_C} \\ MRS(0, x) &= \infty \\ MRS(x, 0) &= 0 \end{aligned}$$

Thus for all positive, finite prices we will have $\frac{p_C}{p_F} \not\geq MRS(0, x)$ and $MRS(x, 0) \not\leq \frac{p_C}{p_F}$, which implies that we will never have a boundary solution, and this is a general property of Cobb-Douglas utility functions. Generally speaking we cannot have a boundary solution when the indifference curves are all bounded away from the axes.

Next we can take first order conditions to solve the consumers problem

which yield the familiar expression

$$\begin{aligned}
 MRS(D_C, D_F) &= \frac{p_C}{p_F} \\
 \frac{aD_F}{bD_C} &= \frac{p_C}{p_F} \\
 \left(\frac{a}{b}\right) p_F D_F &= p_C D_C \\
 \left(\frac{a+b}{b}\right) p_F D_F &= p_C D_C + p_F D_F \\
 &\Rightarrow \frac{p_F D_F}{p_C D_C + p_F D_F} = \frac{b}{a+b}
 \end{aligned}$$

Similarly

$$\begin{aligned}
 p_F D_F &= \left(\frac{b}{a}\right) p_C D_C \\
 &\Rightarrow \frac{p_C D_C}{p_C D_C + p_F D_F} = \frac{a}{a+b}
 \end{aligned}$$

- (b) Let $p = p_C/p_F$. What is the relative demand $RD(p) = D_C/D_F$ at Home?

Solution From above it is

$$\begin{aligned}
 \frac{D_F}{D_C} &= \left(\frac{b}{a}\right) \frac{p_C}{p_F} = \left(\frac{b}{a}\right) p \\
 &\Rightarrow \\
 RD(p) &= \frac{a}{bp}
 \end{aligned}$$

- (c) What is the relative price p^A at Home under autarky?

Solution From lecture, we can consider a representative agent with these preferences. By market clearing, he will consume $D_C = 100$ and $D_F = 200$. The relative price that makes this outcome consistent with demand is

$$\frac{1}{2} = \frac{a}{bp^A} \Rightarrow p^A = 2\frac{a}{b}$$

- (d) Now consider a second economy (Foreign) with same Cobb-Douglas preferences but different aggregate endowments $E_C^* = 200$ and $E_F^* = 100$. What is the relative price p^{A*} abroad under autarky?

Solution The demand is the same so

$$2 = \frac{a}{bp^{A*}} \Rightarrow p^{A*} = \frac{a}{2b}$$

(e) What is the relative price p^T under free trade?

Solution World market clearing will imply that $D_C = D_F = 300$ and preferences are still the same so

$$1 = \frac{a}{bp^T} \Rightarrow p^T = \frac{a}{b}$$

(f) What is the pattern of trade?

Solution Under autarky Home consumes $(D_C, D_F)^A = (100, 200)$ and Foreign consumes $(D_C^*, D_F^*)^A = (200, 100)$. Under free trade, Home's consumption will be given by its relative demand curve and its budget constraint:

$$\begin{aligned} RD(p) &= \frac{a}{bp^T} = 1 \Rightarrow D_C^T = D_F^T \\ p^T D_C^T + D_F^T &= 100p^T + 200 \\ \Rightarrow \\ D_C^T &= D_F^T = \frac{100p^T + 200}{1 + p^T} = 100 \frac{a}{a+b} + 200 \frac{b}{a+b} \\ &= 100\alpha + 200(1 - \alpha) \end{aligned}$$

where $\alpha = \frac{a}{a+b} \in (0, 1)$. Since this is just a weighted average and $a, b > 0$, it follows that $D_C^T = D_F^T \in (100, 200)$. Hence Home exports food and imports cloth. By symmetry, Foreign will consume $D_C^{T*} = D_F^{T*} = 100(1 - \alpha) + 200\alpha$, and it will export food and import cloth.

(g) Show that both countries gain from trade

Solution Since these preferences are rational, suffice it to show that the autarky equilibrium consumption bundles are within the free trade budget set. This is trivially satisfied because Home and Foreign could always afford to just keep their endowments and not trade.

2. Adam and Eve are stranded on a desert island. There are only two goods on the island: Apples (A) and Bananas (B). The utility functions of Adam and Eve are $U^{Adam}(D_A, D_B) = 3D_A + D_B$ and $U^{Eve}(D_A, D_B) = D_A + 3D_B$, respectively. Total endowments on the island are 20 Apples and 60 Bananas. Adam owns all the bananas and Eve all the apples.

(a) Draw the Edgeworth box for this exchange economy, including Adam and Eve's indifference curves and endowments.

Solution Draw your own box here.

(b) Using a graphical analysis, determine the contract curve of this economy.

Solution It will be the right and lower edges of the box assuming Adam's origin is the lower left corner. Draw your own box here.

(c) Will Adam consume any apple in a competitive equilibrium?

Solution The important thing to notice is that each consumer is always at a corner solution unless $p \equiv \frac{p_A}{p_B} = 3$ or $\frac{1}{3}$ in which cases Adam and Eve respectively would be indifferent between all bundles on their budget set. That's because $MRS(D_A, D_B)^A = 3$ and $MRS(D_A, D_B)^E = \frac{1}{3}$. Thus Adam and Eve's demand functions will be piecewise constant and determined by their budget sets. Adam's demand is given by

$$(D_A, D_B)^A = \begin{cases} \left(\frac{60}{p}, 0\right) & p < 3 \\ (D_A, D_B)^A : pD_A^A + D_B^A = 60 & \text{if } p = 3 \\ (0, 60) & p > 3 \end{cases}$$

and Eve's is

$$(D_A, D_B)^E = \begin{cases} (20, 0) & p < \frac{1}{3} \\ (D_A, D_B)^E : pD_A^E + D_B^E = 20p & \text{if } p = \frac{1}{3} \\ (0, 20p) & p > \frac{1}{3} \end{cases}$$

Consider these three cases separately:

Case1: $p < \frac{1}{3}$

This cannot be an equilibrium because nobody wants to consume bananas.

Case2: $p > 3$

This cannot be an equilibrium because nobody wants to consume apples.

Case3: $p \in [\frac{1}{3}, 3]$

$(D_A, D_B)^A = \left(\frac{60}{p}, 0\right)$ and $(D_A, D_B)^E = (0, 20p)$ which is an equilibrium as long as markets clear which just requires that $20p = 60$ and $\frac{60}{p} = 20$ so $p = 3$.

3. There are 2 goods, Cloth (C) and Food (F). Show that if preferences are homothetic, then a consumer with n times the income of another will consume n times more C and F

Solution A consumer with homothetic preferences consumes goods always in the same ratio (taking prices as given). Hence $\frac{D_C}{D_F} = k(p)$ then using the budget constraint

$$pD_C + D_F = w \Rightarrow D_F = \frac{w}{1 + pk(p)} \Rightarrow D_C = \frac{k(p)w}{1 + pk(p)}$$

This displays the result as demand is proportional to income.

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