

**Idiosyncratic Investment (or Entrepreneurial) Risk
in a Neoclassical Growth Model**

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Motivation

empirical importance of entrepreneurial or capital-income risk

~> private businesses account for half of corporate equity, production, and employment

~> typical rich household holds more than half his financial wealth in private equity

~> extreme variation in entrepreneurial returns

~> dramatic lack of diversification

**yet, little research on
uninsurable idiosyncratic investment risk
in the neoclassical growth model**

perhaps more important for business cycles than **labor-income risk**
(**Bewley models**, e.g., Aiyagari 1994, Krusell and Smith 1998)

This paper

~> **uninsurable idiosyncratic investment risk
in a neoclassical growth economy**

~> standard assumptions for preferences and technology:
CRRA and CEIS, neoclassical technology

~> consider both one-sector economy (only private equity)
and a two-sector economy (public and private equity)

~> despite incomplete markets, **closed-form solution**

~> **steady state and transitional dynamics**

~> novel macroeconomic complementarity

Methodological Contribution

diminishing returns at the aggregate level,
but linear returns at the individual level



with CRRA/CEIS preferences
homothetic decision problem (Samuelson-Merton)



linear individual policy rules



wealth distribution irrelevant



closed-form solution for general equilibrium

Findings

~> very different effects than in Bewley models

~> **negative effect on capital and output**

~> in calibrated examples, about **10% loss in output**

~> non-monotonic effect on interest rates

~> pecuniary externality in risk taking

~> dynamic macroeconomic complementarity

~> amplification and persistence

Layout

- The Benchmark Model (only private equity)
- Individual Behavior
- General Equilibrium and Steady State
- The Two-Sector Model (private and public equity)
- Complementarity and Propagation
- Concluding Remarks

The Model

- two inputs (K and L) and a single homogeneous good (Y)
- a continuum of heterogeneous households $i \in [0, 1]$
- competitive product and labor markets

- each household supplies labor in competitive labor market
- each household owns a single firm (family business)
- households can borrow and save in a riskless bond,
but can invest capital only in their own firm
- the firm employs labor from the competitive labor market
- production subject to undiversifiable idiosyncratic risk

Technology and Risks

Output for firm i in period t :

$$y_t^i = F(k_t^i, n_t^i, A_t^i)$$

$F : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ is a neoclassical CRS production technology

A_t^i is an idiosyncratic productivity shock ($F_A, F_{KA}, F_{LA} > 0$ and $\mathbb{E}A = 1$)

Not needed, but useful:

Assumption A1 A is augmented to capital and is lognormally distributed

$\rightsquigarrow \sigma^2 = \text{Var}(\ln A)$ parametrizes incomplete markets

Households

Household capital income = firm profits:

$$\pi_t^i = y_t^i - \omega_t n_t^i = F(k_t^i, n_t^i, A_t^i) - \omega_t n_t^i$$

Budget constraint for household i in period t :

$$c_t^i + k_{t+1}^i + b_{t+1}^i = \omega_t + \pi_t^i + R_t b_t^i$$

Non-negativity constraints: $c_t^i \geq 0$ and $k_t^i \geq 0$

“Natural” borrowing limit:

$$-b_t^i \leq h_t \equiv \sum_{j=1}^{\infty} \frac{q_{t+j} \omega_{t+j}}{q_t}$$

where $q_t = q_{t+1}/R_{t+1}$.

Preferences

Kreps-Porteus/Epstein-Zin non-expected utility:

$$u_t^i = U(c_t^i) + \beta \cdot U\{\mathbb{C}\mathbb{E}_t[U^{-1}(u_{t+1}^i)]\}$$

where

$$\mathbb{C}\mathbb{E}_t(u) \equiv \Upsilon^{-1}[\mathbb{E}_t\Upsilon(u)]$$

U governs intertemporal substitution, Υ governs risk aversion

CEIS and CRRA:

$$U(c) = \frac{c^{1-1/\theta}}{1-1/\theta} \quad \text{and} \quad \Upsilon(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

$\theta > 0$ elasticity of intertemporal substitution

$\gamma > 0$ degree of relative risk aversion

Equilibrium

Definition A competitive equilibrium is a deterministic sequence of prices $\{R_t, \omega_t\}_{t=0}^{\infty}$ and a collection of contingent individual plans $\{c_t^i, n_t^i, k_{t+1}^i, b_{t+1}^i\}_{t=0}^{\infty}$, $i \in [0, 1]$, such that:

- (i) The plan $\{c_t^i, n_t^i, k_{t+1}^i, b_{t+1}^i\}_{t=0}^{\infty}$ is optimal for all i .
- (ii) The labor market clears in every period: $\int n_t^i = 1$.
- (iii) The bond market clears in every period: $\int b_t^i = 0$.

Remark: in open economy $\rightarrow R$ exogenous

Optimal Individual Behavior

By CRS,

$$\frac{\pi_t^i}{k_t^i} = F\left(1, \frac{n_t^i}{k_t^i}, A_t^i\right) - \omega_t \frac{n_t^i}{k_t^i}$$

It follows that

Proposition 1 Labor demand and capital income are decreasing in A , decreasing in ω , and linear in k

$$n_t^i = n(A_t^i, \omega_t) \cdot k_t^i \quad \text{and} \quad \pi_t^i = r(A_t^i, \omega_t) \cdot k_t^i$$

where $r(A, \omega) \equiv \max_L [F(1, L, A) - \omega L]$ and $n(A, \omega) \equiv \arg \max_L [.]$

Define financial wealth as

$$w_t^i \equiv \omega_t + \pi_t^i + R_t b_t^i$$

By Proposition 1,

$$w_t^i = \omega_t + r(A_t^i, \omega_t)k_t^i + R_{t-1}b_t^i$$

Given $\{R_t, \omega_t\}_{t=0}^{\infty}$, the value function $V_t(w)$ solves

$$V_t(w_t^i) = \max_{(c_t^i, k_{t+1}^i, b_{t+1}^i)} U(c_t^i) + \beta \cdot U\Upsilon^{-1} \{ \mathbb{E}_t[\Upsilon U^{-1} V_{t+1}^i(w_{t+1}^i)] \}$$

subject to

$$c_t^i + k_{t+1}^i + b_{t+1}^i = w_t^i$$

$$w_{t+1}^i = \omega_{t+1} + r(A_{t+1}^i, \omega_{t+1})k_{t+1}^i + R_{t+1}b_{t+1}^i$$

$$c_t^i \geq 0 \quad k_{t+1}^i \geq 0 \quad -b_{t+1}^i \leq h_{t+1}$$

Individual Savings and Investment

Proposition 2 The optimal individual path satisfies

$$w_t^i = \omega_t + r(A_t^i, \omega_t)k_t^i + R_t b_t^i$$

$$c_t^i = (1 - s_t)(w_t^i + h_t)$$

$$k_{t+1}^i = s_t \phi_t (w_t^i + h_t)$$

$$b_{t+1}^i = s_t (1 - \phi_t)(w_t^i + h_t) - h_{t+1}$$

where

$$\phi_t = \phi(\omega_{t+1}, R_{t+1}) = \arg \max_{\phi} \left\{ \int_A [\phi \cdot r(A, \omega_{t+1}) + (1 - \phi)R_{t+1}]^{1-\gamma} \right\}^{\frac{1}{1-\gamma}}$$

$$\rho_t = \rho(\omega_{t+1}, R_{t+1}) = \max_{\phi} \left\{ \int_A [\phi \cdot r(A, \omega_{t+1}) + (1 - \phi)R_{t+1}]^{1-\gamma} \right\}^{\frac{1}{1-\gamma}}$$

$$s_t = \left[1 + \left(\sum_{s=t}^{\infty} \prod_{\tau=t}^s \beta^{\theta} \rho_{\tau}^{\theta-1} \right)^{-1} \right]^{-1}$$

Lemma Under A1,

$$\phi_t \approx \frac{\ln \mu_{t+1}}{\gamma \sigma^2} \quad \text{and} \quad \ln \rho_t \approx \ln R_{t+1} + \frac{(\ln \mu_{t+1})^2}{\gamma \sigma^2}$$

where

$$\mu_{t+1} = \frac{f'(K_{t+1})}{R_{t+1}} \quad \text{and} \quad \sigma^2 = \text{Var}[\ln A].$$

General Equilibrium

Linear policy rules \Rightarrow **wealth distribution irrelevant**

Aggregates satisfy

$$N_t = \bar{n}(\omega_t)K_t$$

$$\Pi_t = \bar{r}(\omega_t)K_t$$

$$\Pi_t + \omega_t N_t = f(K_t)$$

where $\bar{n}(\omega) \equiv \int_A n(A, \omega)$, $\bar{r}(\omega) \equiv \int_A r(A, \omega)$, and $f(K) = F(K, 1, \bar{A})$

General Equilibrium

Proposition 3 The equilibrium path $\{C_t, K_t, H_t, \omega_t, R_t\}_{t=0}^{\infty}$ satisfies

$$C_t + K_{t+1} = f(K_t)$$

$$C_t = (1 - s_t)[f(K_t) + H_t]$$

$$K_{t+1} = \phi_t s_t [f(K_t) + H_t]$$

$$H_t = \frac{1}{R_{t+1}} [\omega_{t+1} + H_{t+1}]$$

$$\bar{n}(\omega_t) K_t = 1$$

$$1 - s_t = \frac{1}{1 + \beta^\theta \rho_t^{\theta-1} (1 - s_{t+1})^{-1}}$$

where $\phi_t = \phi(\omega_{t+1}, R_{t+1})$ and $\rho_t = \rho(\omega_{t+1}, R_{t+1})$.

Steady State

Proposition 4 In steady state, ϕ_∞ , K_∞ and R_∞ solve

$$\mathbb{E}\{[\phi_\infty A f'(K_\infty) + (1 - \phi_\infty)R_\infty]^{-\gamma} [f'(K_\infty) - R_\infty]\} = 0$$

$$\beta^\theta \left\{ \mathbb{E}[\phi_\infty A f'(K_\infty) + (1 - \phi_\infty)R_\infty]^{1-\gamma} \right\}^{\frac{\theta-1}{1-\gamma}} \times [\phi_\infty f'(K_\infty) + (1 - \phi_\infty)R_\infty] = 1$$

$$\frac{f(K_\infty) - f'(K_\infty)K_\infty}{(R_\infty - 1)K_\infty} = \frac{1 - \phi_\infty}{\phi_\infty}$$

Proposition 5 Under A1,

$$\ln f'(K_\infty) \approx \ln R_\infty + \sigma \sqrt{\frac{2\gamma\theta}{1+\theta} \ln(\beta R_\infty)^{-1}}$$

\Rightarrow **idiosyncratic risk necessarily reduces the capital stock for any given interest rate**

Calibration

$$\gamma = 2$$

$$\theta = 1$$

$$\beta^{-1} - 1 = 5\%$$

$$\alpha = 40\%$$

$$\delta = 5\%$$

σ = standard deviation of investment return

$$\sigma = 50\% \quad \text{or} \quad \sigma = 25\%$$

Numerical Simulations

σ	γ	θ	$\beta^{-1}-1$	α	δ	Output Loss	Capital Loss	Interest Rate	Private Premium
50%	2	1	5%	40%	10%				
20%									
50%	1	1	5%	40%	10%				
20%									
50%	3	1	5%	40%	10%				
20%									
50%	2	2	5%	40%	10%				
20%									
50%		.5							
20%									
50%	2	1	8%	40%	10%				
20%									
50%			5%	60%	10%				
20%									
50%			5%	40%	5%				
20%									

Table 1

Consider the case that private equity accounts for all capital and production. The table reports the impact of idiosyncratic investment risk on income, savings, and interest rates for a series of calibrations. The chosen parameter values are in the first six columns, and the implied effects in the last four columns. “Output loss” and “capital loss” refer to the percentage reduction in the steady-state level of output and capital as compared to complete markets. “Interest rate” is the rate of return in riskless bonds, while “private premium” is the excess return earned in private equity.

Two Sectors: Private and Public Equity

Public equity = no idiosyncratic risk

Let X_t and L_t denote capital and labor in public equity; output is

$$G(X_t, L_t)$$

where G is a neoclassical production function

Assumption A2 For $\mu > 1$,

$$G(X, L) = F(X, L, \bar{A}/\mu)$$

$\leadsto \mu$ pins down the private equity premium when both sectors are open

Individual Behavior

The household budget:

$$c_t^i + k_{t+1}^i + x_{t+1}^i + b_{t+1}^i \leq r(A_t^i, \omega_t)k_t^i + R_t x_t^i + R_t b_t^i + \omega_t N^i.$$

where x_{t+1}^i denotes investment in public equity

The optimal plan satisfies

$$w_t^i = r(A_t^i, \omega_t)k_t^i + R_t x_t^i + R_t b_t^i + \omega_t N^i$$

$$c_t^i = (1 - s_t)(w_t^i + h_t^i)$$

$$k_{t+1}^i = s_t \phi_t (w_t^i + h_t^i)$$

$$x_{t+1}^i + b_{t+1}^i = s_t (1 - \phi_t) (w_t^i + h_t^i) - h_t^i$$

where ϕ_t , ρ_t , and s_t are defined as before.

General Equilibrium

By profit maximization,

$$L_t = l(\omega_t)X_t \quad \text{and} \quad R_t = R(\omega_t)$$

where $R(\omega) \equiv \max_L [G(1, L) - \omega L]$ and $l(\omega) \equiv \arg \max_L []$.

Lemma Under A1 and A2,

$$\bar{n}(\omega) = \mu l(\omega) \quad \text{and} \quad \bar{r}(\omega) = \mu R(\omega)$$

It follows that

$$\phi_t = \phi \approx \frac{\ln \mu}{\gamma \sigma^2} \quad \text{and} \quad \frac{\rho_t}{R_{t+1}} = \varrho \approx \exp\left(\frac{\ln \mu}{2\gamma \sigma^2}\right)$$

General Equilibrium

Proposition 6 In any equilibrium in which both sectors are active, the equilibrium dynamics satisfy

$$C_t + K_{t+1} + X_{t+1} = W_t = F(K_t, \bar{n}(\omega_t)K_t, \bar{A}) + G(X_t, l(\omega_t)X_t)$$

$$C_t = (1 - s_t)[W_t + H_t]$$

$$K_{t+1} = \phi_t s_t [W_t + H_t]$$

$$H_t = \frac{1}{R_{t+1}} [\omega_{t+1} + H_{t+1}]$$

$$\bar{n}(\omega_t)K_t + l(\omega_t)X_t = 1$$

$$R_t = R(\omega_t)$$

$$(1 - s_t) = \frac{1}{1 + \beta^\theta (\rho_t)^{\theta-1} (1 - s_{t+1})^{-1}}$$

where $\phi_t = \phi \approx \frac{\ln \mu}{\gamma \sigma^2}$, $\rho_t = \varrho R_{t+1}$, $\varrho \approx \exp\left(\frac{\ln \mu}{2\gamma \sigma^2}\right)$.

Steady State

Proposition 7 A steady state in which both sectors are active is unique whenever it exists, and it exists if and only if σ is sufficiently high. The steady state satisfies

$$(\beta R_\infty)^\theta \varrho^{\theta-1} (\phi\mu + 1 - \phi) = 1$$

$$R(\omega_\infty) = R_\infty$$

$$K_\infty = \frac{1/l(\omega_\infty) + \omega_\infty/(R_\infty - 1)}{\mu + 1/\phi - 1}$$

$$X_\infty = 1/l(\omega_\infty) - \mu K_\infty$$

Proposition 7 There exists $\underline{\theta} < 1$ such that, whenever $\theta > \underline{\theta}$, an increase in σ raises R , has an ambiguous effect on $K + X$, but necessarily reduces K , Y , C , Y/L and Y/K .

Numerical Simulations

σ	γ	θ	$\beta^{-1}-1$	α	δ	Output Loss	Capital Loss	Interest Rate	Private Premium
50%	2	1	5%	40%	10%				
20%									
50%	1	1	5%	40%	10%				
20%									
50%	3	1	5%	40%	10%				
20%									
50%	2	2	5%	40%	10%				
20%									
50%		.5							
20%									
50%	2	1	8%	40%	10%				
20%									
50%			5%	60%	10%				
20%									
50%			5%	40%	5%				
20%									

Table 2

Consider the case that risk-free public equity coexists with risky private equity. The table reports the impact of idiosyncratic investment risk on income, savings, and interest rates for a series of calibrations. “Output loss” and “capital loss” now refer to the combined output and capital in private and public equity. The “interest rate” is the rate of return in either riskless bonds or public equity. The “private premium” is pinned down by the technological parameter μ and is calibrated so that private and public equity each account for half of the aggregate capital stock.

Propagation Mechanism

Along the transition,

$$K_{t+1} = \phi_t s_t [f(K_t) + H_t]$$
$$H_t = \sum_{j=1}^{\infty} \frac{q_{t+j}}{q_t} \omega(K_{t+j}).$$

Hence

$\{K_{t+1}, K_{t+2}, \dots\}$ increases with $\{H_t, H_{t+1}, \dots\}$
 $\{H_t, H_{t+1}, \dots\}$ increases with $\{K_{t+1}, K_{t+2}, \dots\}$

↔ **dynamic macroeconomic complementarity**

↔ amplification and persistence

Remarks on Propagation Mechanism

- ✓ a general-equilibrium phenomenon
- ✓ derives from a pecuniary externality

✓ relies on two premises:

- (1) investment subject to undiversifiable idiosyncratic risk
- (2) risk taking sensitive to anticipated future economic activity

(1) absent from Aiyagari (1994), Krusell and Smith (1998);

(2) absent from Bernanke and Gertler (1989, 1990), Kiyotaki and Moore (1997),
as well as Bencivenga and Smith (1991), Obstfeld (1994), Krebs (2003).

Numerical Simulations

Concluding Remarks/Future Research

- lower capital and output in the steady state
- amplification and persistence in transitional dynamics
- pecuniary externality \rightsquigarrow inefficiency? coordination failure?
- stabilization policy? optimal taxation?
- welfare cost of business cycles?
- wealth distribution?
- quantitative analysis (back to Krusell and Smith)