14.462 Advanced Macroeconomics

Spring 2004

Problem Set 3 Solution

1. The differential equation in the case of one good is $\dot{Q}_1 = \Phi_1(Q_1)$ where

$$\Phi_1(Q_1) = \delta_1 \left[1 - F(a_0 + A(Q_1)) - Q_1 \right].$$

Here we are given a specific distribution:

$$F(x) = \min \left[\max \left[\frac{x - (\bar{y} - \sigma)}{2\sigma}, 0 \right], 1 \right]$$

and so the condition for the steady state is

$$Q_1 = 1 - \min \left[\max \left[\frac{a_0 + A(Q_1) - (\bar{y} - \sigma)}{2\sigma}, 0 \right], 1 \right]$$

If $a_0 + A(1) < \bar{y} - \sigma$, then $Q_1 = 1$ is a steady state: if cumulative output is equal to one, then productivity is sufficiently high such that everybody can afford to buy the good. This steady state is also stable: for Q_1 slightly below 1, we have $\dot{Q}_1 > 0$. If $a_0 + A(0) > \bar{y} - \sigma$, then $Q_1 = 0$ is a steady state: with no cumulative output, productivity is so low that nobody can afford to buy the good. If $a_0 + A(Q_1) \in [\bar{y} - \sigma, \bar{y} + \sigma]$, then Q_1 is a steady state if

$$Q_1 = \frac{1}{2} - \frac{a_0 + A(Q_1) - \bar{y}}{2\sigma}$$

2. We already discussed stability for the corner-solutions, so let's focus on the stability of possible steady states with $a_0 + A(Q_1) \in [\bar{y} - \sigma, \bar{y} + \sigma]$. Stability requires that a small increase in Q_1 makes $\Phi(Q_1)$ negative and that a small decrease makes it positive. This leads to the condition:

$$A'(Q_1) > -2\sigma.$$

It is clear that the condition $A'(Q_1) < 0$ is a typo, both in terms of the sign of the inequality and the right hand side. The destabilizing force is that an increase in cumulative output increases productivity which increases demand which in turn increases cumulative output. For a steady state to be stable, this force cannot be to strong. This force is weaker if $A'(Q_1)$ is not too negative, because then an increase in cumulative output does not reduce productivity that much. The force is also weaker if σ is large, because then a given reduction in productivity does not increase demand very much.

3. From now on let's focus on the interesting case in which $a_0 + A(Q_1) \in (\bar{y} - \sigma, \bar{y} + \sigma)$ and in which the stability condition is satisfied. Implicit differentiation yields

$$\frac{\partial Q_1}{\partial a_0} = -\frac{1}{2\sigma + A'(Q_1)}.$$

Making food more difficult to produce reduces steady state cumulative output of good one.

4. Now we get

$$\frac{\partial Q_1}{\partial \sigma} = \frac{1}{2\sigma + A'(Q_1)} \frac{a_0 + A(Q_1) - \bar{y}}{\sigma}$$

Thus an increase in inequality will increase steady state cumulative output if $\bar{y} < a_0 + A(Q_1)$, that is if the average agent cannot afford the industrial good. In this case only the upper tail can afford the industrial good, and increasing inequality puts more mass into the upper tail. If instead $\bar{y} > a_0 + A(Q_1)$, then only the lower tail cannot afford the industrial good. Now increasing inequality increases the number of people unable to afford the industrial good by putting more mass in the lower tail.

5. I guess here one may want to distinguish between cumulative output and output (in steady state they are the same, but they can be different during the transition), and the path of output is perhaps more interesting. Suppose we are in the situation where an increase in inequality increases the steady state level of cumulative output. Of course cumulative output cannot jump and will converge gradually to the new steady state. However, output will jump up immediately as more people can buy the industrial good, and it continues to increase during the transition as the production of the industrial good becomes cheaper as time passes.

6. The analogous steady state condition for the second industrial good is

$$Q_2 = \frac{1}{2} - \frac{a_0 + A_1(Q_1) + A_2(Q_2) - \bar{y}}{2\sigma}$$

and implicit differentiation yields

$$\frac{\partial Q_2}{\partial \sigma} = \frac{\frac{a_0 + A(Q_2) + A_2(Q_2) - \bar{y}}{\sigma} - A_1'(Q_1)\frac{\partial Q_1}{\partial \sigma}}{2\sigma + A_1'(Q_1)}$$

and substituting our earlier result gives

$$\frac{\partial Q_2}{\partial \sigma} = \frac{\frac{a_0 + A(Q_2) + A_2(Q_2) - \bar{y}}{\sigma} - A_1'(Q_1)\frac{1}{2\sigma + A_1'(Q_1)}\frac{a_0 + A_1(Q_1) - \bar{y}}{\sigma}}{2\sigma + A'(Q_1)}$$

If $\bar{y} \leq a_0 + A_1(Q_1)$, then inequality clearly increases output of the second industrial good. Holding Q_1 constant the same logic as in the previous part applies, and the

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increase in Q_1 goes in the same direction (the logic is the same as that of a fall in a_0). If $\bar{y} \geq a_0 + A_1(Q_1) + A(Q_2)$, inequality clearly reduces output, as now both effects are reversed. But the effect is ambiguous for intermediate values of $\bar{y}$. The direct effect of the increase in inequality is positive, but the fall in Q_1 works in the opposite direction. It could happen that Q_2 increases while Q_1 falls. So it need not be the case that all industrial goods move in the same direction.