14.30 Introduction to Statistical Methods in Economics
Spring 2009

Problem Set #6 - Solution

14.30 - Intro. to Statistical Methods in Economics

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Question One

Let X be a random variable that is uniformly distributed on $[0,1]$ (i.e. $f(x)=1$ on that interval and zero elsewhere). In Problem Set #4, you use the "2-step"/CDF technique and the transformation method to determine the PDF of each of the following transformations, $Y=g(X)$. Now that you have the PDFs, compute (a) $\mathbb{E}[g(X)]$, (b) $g(\mathbb{E}[X])$, (c) $Var(g(X))$ and (d) $g(Var(X))$ for each of the following transformations:

1. $Y=X^{\frac{1}{4}}$, $f_Y(y)=4y^3$ on $[0,1]$ and zero otherwise.

   - Solution to 1: We compute the four components:

     (a) $\mathbb{E}[g(X)]=\int_0^1 y(4y^3)dy=(\frac{4}{5}y^5)_0^1=\frac{4}{5}=0.80$ or we can compute it using X: $\mathbb{E}[g(X)]=\int_0^1 x^{\frac{1}{4}}dx=(\frac{4}{5}x^{\frac{5}{4}})_0^1=\frac{4}{5}$.

     (b) $g(\mathbb{E}[X])=\left(\int_0^1 x dx\right)^{\frac{1}{4}}=\left(\frac{1}{2}\right)^{\frac{1}{4}}=\frac{1}{\sqrt[4]{2}}=0.84$.

     (c) The variance uses the result in part (a)

$$\begin{aligned} Var(g(X)) &= \int_0^1 (y-\frac{4}{5})^2(4y^3)dy \\ &= 4\int_0^1 (y^5-\frac{8}{5}\cdot y^4+\frac{16}{25}\cdot y^3)dy \\ &= 4\left(\frac{1}{6}y^6-\frac{8}{25}\cdot y^5+\frac{4}{25}\cdot y^4\right)_0^1 \\ &= 4\left(\frac{1}{6}-\frac{8}{25}+\frac{4}{25}\right)=4\left(\frac{25}{150}-\frac{24}{150}\right) \\ Var(g(X)) &= \frac{2}{75}=0.02667 \end{aligned}$$

     (d) We need to compute $Var(X)$ first:

$$\begin{aligned} Var(X) &= \int_0^1 (x-\frac{1}{2})^2 dx \\ &= \int_0^1 (x^2-x+\frac{1}{4})dx \end{aligned}$$

$$
\begin{aligned}
&= [\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x]_0^1 \\
&= \frac{1}{3} - \frac{1}{2} + \frac{1}{4} \\
Var(X) &= \frac{1}{12}
\end{aligned}
$$

And then transform it: $g(Var(X)) = \left(\frac{1}{12}\right)^{\frac{1}{4}} = \frac{1}{\sqrt[4]{12}} = 0.537$

2. $Y = e^{-X}$, $f_Y(y) = \frac{1}{y}$ on $[\frac{1}{e}, 1]$ and zero otherwise.

- Solution to 2: We compute the four components:
  (a) $\mathbb{E}[g(X)] = \int_{\frac{1}{e}}^{1} y(\frac{1}{y})dy = (y)_0^1 = 1 - \frac{1}{e} = 0.632$ or we can compute it using X: $\mathbb{E}[g(X)] = \int_0^1 e^{-x}dx = (-e^{-x})_0^1 = -\frac{1}{e} + 1$.
  (b) $g(\mathbb{E}[X]) = e^{-\left(\int_0^1 x dx\right)} = e^{-\left(\frac{1}{2}\right)} = 0.607$.
  (c) The variance uses the result in part (a), $\bar{y} \equiv \mathbb{E}[Y] = 1 - \frac{1}{e}$,

$$
\begin{aligned}
Var(g(X)) &= \int_{\frac{1}{e}}^{1} (y-\bar{y})^2 \frac{1}{y} dy \\
&= \int_{\frac{1}{e}}^{1} \left(y - 2\bar{y} + \bar{y}^2 \cdot \frac{1}{y}\right) dy \\
&= \left(\frac{1}{2}y^2 - 2\bar{y} \cdot y + \bar{y}^2 \log y\right)_{\frac{1}{e}}^{1} \\
&= \left(\frac{1}{2} - 2\left(1 - \frac{1}{e}\right) - \left(\frac{1}{2}\frac{1}{e^2} - 2\left(1 - \frac{1}{e}\right) \cdot \frac{1}{e} - \left(1 - \frac{1}{e}\right)^2\right)\right) \\
&= \left(\frac{1}{2} - 2 + \frac{2}{e} - \frac{1}{2}\frac{1}{e^2} + \frac{2}{e} - \frac{2}{e^2} + 1 - \frac{2}{e} + \frac{1}{e^2}\right) \\
&= -\frac{1}{2}\left(1 - 4\frac{1}{e} + 3\frac{1}{e^2}\right) \\
&= -\frac{1}{2}(1 - \frac{1}{e})(1 - 3\frac{1}{e}) \\
Var(g(X)) &= 0.033
\end{aligned}
$$

  (d) Using $Var(X)$ from part (a), $Var(X) = \frac{1}{12}$, we apply $g(\cdot)$: $g(Var(X)) = e^{-\frac{1}{12}} = 0.920$.

3. $Y = 1 - e^{-X}$, $f_Y(y) = \frac{1}{1-y}$ on $[0, 1 - \frac{1}{e}]$ and zero otherwise.

- Solution to 2: We compute the four components:

(a) We need to do a little more algebra for this problem:

$$\begin{aligned}
\mathbb{E}[g(X)] &= \int_0^{1-\frac{1}{e}} y(\frac{1}{1-y})dy \\
&= \int_0^{1-\frac{1}{e}} (\frac{y}{1-y} + \frac{1-y}{1-y} - 1)dy \\
&= \int_0^{1-\frac{1}{e}} (\frac{1}{1-y} - 1)dy \\
&= (-\log(1-y) - y)_0^{1-\frac{1}{e}} \\
\mathbb{E}[g(X)] &= 1 - 1 + \frac{1}{e} = \frac{1}{e} = 0.368
\end{aligned}$$

or we can compute it using X: $\mathbb{E}[g(X)] = \int_0^1 (1-e^{-x})dx = (x + e^{-x})_0^1 = 1 + \frac{1}{e} - 1 = \frac{1}{e}$.

(b) $g(\mathbb{E}[X]) = 1 - e^{-\left(\int_0^1 x dx\right)} = 1 - e^{-\left(\frac{1}{2}\right)} = 0.393$.

(c) The variance uses the result in part (a), $\bar{y} \equiv \mathbb{E}[Y] = \frac{1}{e}$, combined with one of the identities for the variance:

$$\begin{aligned}
Var(g(X)) &= \mathbb{E}[g(X)^2] - \mathbb{E}[g(X)]^2 \\
&= \int_0^{1-\frac{1}{e}} y^2 \frac{1}{1-y} dy - \frac{1}{e^2} \\
&= \int_0^{1-\frac{1}{e}} y\left(\frac{1}{1-y} - 1\right) dy - \frac{1}{e^2} \\
&= \int_0^{1-\frac{1}{e}} \left(\frac{y}{1-y} - y + \frac{1-y}{1-y} - 1\right) dy - \frac{1}{e^2} \\
&= \int_0^{1-\frac{1}{e}} \left(-y + \frac{1}{1-y} - 1\right) dy - \frac{1}{e^2} \\
&= \left(-\frac{1}{2}y^2 - \log(1-y) - y\right)_0^{1-\frac{1}{e}} - \frac{1}{e^2} \\
Var(g(X)) &= 0.0328
\end{aligned}$$

(d) Using $Var(X)$ from part (a), $Var(X) = \frac{1}{12}$, we apply $g(\cdot)$: $g(Var(X)) = 1 - e^{-\frac{1}{12}} = 0.080$.

4. How does (a) $\mathbb{E}[g(X)]$ compare to (b) $g(\mathbb{E}[X])$ and (c) $Var(g(X))$ to (d) $g(Var(X))$ for each of the above transformations? Are there any generalities that can be noted? Explain.

- Solution to 1: The table below gives the comparisons:

	(a)	(b)	(c)	(d)
$X^{\frac{1}{4}}$	0.80	0.84	0.027	0.537
e^{-X}	0.632	0.607	0.033	0.920
$1-e^{-X}$	0.368	0.393	0.033	0.080

What we see is that concave functions like $X^{\frac{1}{4}}$ and $1-e^{-X}$ have $\mathbb{E}[g(X)] < g(\mathbb{E}[X])$ and convex functions have $\mathbb{E}[g(X)] > g(\mathbb{E}[X])$. This is just an example of Jensen's inequality, that except for linear $g(\cdot)$, $\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])$. An extension of Jensen's inequality for the variance would be to define $h(x) = (g(x) - \mathbb{E}[g(x)])^2$ and determine whether $h(\cdot)$ is concave or convex, depending on the concavity of $g(\cdot)$. Well, a quick derivation yields:

$$\begin{aligned}
\frac{\partial}{\partial x} h(x) &= 2(g(x) - \mathbb{E}[g(x)])g'(x) \\
\frac{\partial^2}{\partial x^2} h(x) &= 2(\underbrace{g'(x)g'(x)}_{+} - \underbrace{\mathbb{E}[g(x)]}_{+/-}\underbrace{g''(x)}_{+/-}
\end{aligned}$$

So, basically, the concavity is completely ambiguous as it depends upon $\mathbb{E}[g(x)]$ which can be positive or negative for any function. Thus, we generally can't say anything about how $g(Var(X))$ should compare to $Var(g(X))$, except, perhaps that they're generally not equal, even though we can't sign the bias, unless, of course, $g(\cdot)$ is linear. Then we are guaranteed to have a convex function, as $\frac{\partial^2}{\partial x^2}h(x) > 0$ since the second term is zeroed out.

Question Two

Compute the expectation and the variance for each of the following PDF's.

1. $f_X(x) = ax^{a-1}$, $0 < x < 1$, $a > 0$.

   - Solution to 1: We first compute the expectation.

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^1 x \cdot ax^{a-1} dx \\
&= \int_0^1 ax^a dx \\
&= \frac{a}{a+1} x^{a+1}|_0^1 \\
\mathbb{E}[X] &= \frac{a}{a+1}
\end{aligned}$$

   Now, we compute the variance.

$$\begin{aligned}
Var(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \int_0^1 x^2 \cdot ax^{a-1} dx - \mathbb{E}[X]^2 \\
&= \int_0^1 ax^{a+1} dx - \mathbb{E}[X]^2 \\
&= \frac{a}{a+2} x^{a+2}|_0^1 - \left(\frac{a}{a+1}\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a}{a+2} - \left(\frac{a}{a+1}\right)^2 \\
Var(X) &= \frac{a}{(a+2)(a+1)^2}
\end{aligned}
$$

2. $f_X(x) = \frac{1}{n}$, $x = 1, 2, \ldots, n$, where n is an integer.

   - Solution to 2: We first compute the expectation.

$$
\begin{aligned}
\mathbb{E}[X] &= \sum_{x=1}^{n} \frac{1}{n} x \\
&= \frac{1}{n} \sum_{x=1}^{n} x \\
&= \frac{1}{n} \frac{n(n+1)}{2} \\
\mathbb{E}[X] &= \frac{n+1}{2}
\end{aligned}
$$

   And then, we compute the variance. We need to know the sum of the finite series $\sum_{x=1}^{n} x^2$ (there are many clever ways to compute this, or you can find it at http://en.wikipedia.org/wiki/List_of_mathematical_series).

$$
\begin{aligned}
Var(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \sum_{x=1}^{n} \frac{1}{n} x^2 - \left(\frac{n+1}{2}\right)^2 \\
&= \frac{1}{n} \sum_{x=1}^{n} x^2 - \left(\frac{n+1}{2}\right)^2 \\
&= \frac{1}{n} \frac{n(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 \\
&= (n+1)\left(\frac{2n+1}{6} - \frac{3n+3}{12}\right) \\
&= \frac{(n+1)(n-1)}{12} = \frac{1}{12}(n^2-1) \\
Var(X) &= \frac{1}{12}(n^2-1)
\end{aligned}
$$

3. $f_X(x) = \frac{3}{2}(x-1)^2$, $0 < x < 2$.

   - Solution to 3: Compute the expecation.

$$
\begin{aligned}
\mathbb{E}[X] &= \int_0^2 x \cdot \frac{3}{2}(x-1)^2 dx \\
&= \frac{3}{2} \int_0^2 (x^3 - 2x^2 + x) dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{3}{2}(\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2)_0^2 \\
&= \frac{3}{2}(\frac{1}{4}16 - \frac{2}{3}8 + \frac{1}{2}4) \\
\mathbb{E}[X] &= 1
\end{aligned}
$$

And now compute the variance.

$$
\begin{aligned}
Var(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \int_0^2 x^2 \cdot \frac{3}{2}(x-1)^2 dx - 1 \\
&= \frac{3}{2}\int_0^2 (x^4 - 2x^3 + x^2) dx - 1 \\
&= \frac{3}{2}(\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3)_0^2 - 1 \\
&= \frac{3}{2}(\frac{1}{5}32 - \frac{1}{2}16 + \frac{1}{3}8) - 1 \\
Var(X) &= \frac{3}{5}
\end{aligned}
$$

Question Three

Suppose that X, Y, and Z are independently and identically distributed with mean zero and variance one. Calculate the following:

1. $\mathbb{E}[3X + 2Y + Z]$

   - Solution to 1: $\mathbb{E}[3X + 2Y + Z] = 3\mathbb{E}[X] + 2\mathbb{E}[Y] + \mathbb{E}[Z] = 3 \cdot 0 + 2 \cdot 0 + 1 \cdot 0 = 0$.

2. $Var[5X - 3Y - 2Z]$

   - Solution to 2:

$$
\begin{aligned}
Var[5X - 3Y - 2Z] &= Var(5X) + Var(-3Y) + Var(-2Z) \\
&= 25Var(X) + 9Var(Y) + 4Var(Z) \\
&= 25 \cdot 1 + 9 \cdot 1 + 4 \cdot 1 \\
Var[5X - 3Y - 2Z] &= 38
\end{aligned}
$$

3. $Cov[X - Y + 4, 2X + 3Y + Z]$

   - Solution to 3:

$$
\begin{aligned}
Cov[X - Y + 4, 2X + 3Y + Z] &= Cov[X - Y, 2X + 3Y + Z] \\
&= Cov[X, 2X] + Cov(X, 3Y) + Cov(X, Z) \\
&\quad + Cov(-Y, 2X) + Cov(-Y, 3Y) + Cov(-Y, Z)
\end{aligned}
$$

$$
\begin{aligned}
&= 2Var(X) + 3 \cdot Cov(X, Y) + Cov(X, Z) \\
&\quad -2 \cdot Cov(Y, X) - 3 \cdot Var(Y) - Cov(Y, Z) \\
&= 2 \cdot 1 + 3 \cdot 0 + 1 \cdot 0 - 2 \cdot 0 - 3 \cdot 1 - 1 \cdot 0 \\
Cov[X - Y + 4, 2X + 3Y + Z] &= -1
\end{aligned}
$$

4. $E[3XY]$

   - Solution to 4:

$$
\begin{aligned}
E[3XY] &= \int_{x \in X} \int_{y \in Y} 3xyf(x, y)dydx \\
&= 3 \int_{x \in X} \int_{y \in Y} xyf_X(x)f_Y(y)dydx \\
&= 3 \int_{x \in X} xf_X(x)dx \int_{y \in Y} yf_Y(y)dy \\
&= 3\mathbb{E}[X]\mathbb{E}[Y] = 3 \cdot 0 \cdot 0 \\
E[3XY] &= 0
\end{aligned}
$$

Question Four

Simplify the following expressions for random variables X and Y and scalar constants $a, b \in \mathbb{R}$:

1. $Var(aX + b)$

   - Solution to 1: $Var(aX + b) = Var(aX) = a^2Var(X)$.

2. $Cov(aX + c, bY + d)$

   - Solution to 2: $Cov(aX + c, bY + d) = Cov(aX, bY) = abCov(X, Y)$.

3. $Var(aX + bY)$

   - Solution to 3:

$$
\begin{aligned}
Var(aX + bY) &= Var(aX) + Var(bY) + 2Cov(aX, bY) \\
&= a^2Var(X) + b^2Var(Y) + 2abCov(X, Y)
\end{aligned}
$$

Question Five

(Bain/Engelhardt p.190)

Suppose X and Y are continuous random variables with joint PDF $f(x, y) = 4(x - xy)$ if $0 < x < 1$ and $0 < y < 1$, and zero otherwise.

1. Find $\mathbb{E}[X^2Y]$.

   - Solution to 1:

$$\begin{aligned}
\mathbb{E}[X^2Y] &= \int_0^1 \int_0^1 x^2y \cdot 4(x - xy)dydx \\
&= \int_0^1 \int_0^1 4(x^3y - x^3y^2)dydx \\
&= \int_0^1 4(\frac{1}{2}x^3y^2 - \frac{1}{3}x^3y^3)_0^1 dx \\
&= \int_0^1 4(\frac{1}{2}x^3 - \frac{1}{3}x^3)dx \\
&= \int_0^1 \frac{2}{3}x^3 dx = \frac{2}{12}x^4|_0^1 \\
\mathbb{E}[X^2Y] &= \frac{1}{6}
\end{aligned}$$

2. Find $\mathbb{E}[X - Y]$.

   - Solution to 2:

$$\begin{aligned}
\mathbb{E}[X - Y] &= \mathbb{E}[X] - \mathbb{E}[Y] \\
&= \int_0^1 \int_0^1 (x - y) \cdot 4(x - xy)dydx \\
&= 4\int_0^1 \int_0^1 (x^2 - x^2y - xy + xy^2)dydx \\
&= 4\int_0^1 (x^2y - \frac{1}{2}x^2y^2 - \frac{1}{2}xy^2 + \frac{1}{3}xy^3)_0^1 dx \\
&= 4\int_0^1 (x^2 - \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{3}x)dx \\
&= 4(\frac{1}{3}x^3 - \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{6}x^2)_0^1 \\
&= 4(\frac{1}{3} - \frac{1}{6} - \frac{1}{4} + \frac{1}{6}) \\
&= \frac{1}{3}
\end{aligned}$$

   If we look carefully, we can see what the $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ are:

$$\begin{aligned}
\mathbb{E}[X] &= 4(\frac{1}{3} - \frac{1}{6}) = \frac{2}{3} \\
\mathbb{E}[Y] &= 4(\frac{1}{4} - \frac{1}{6}) = \frac{1}{3}
\end{aligned}$$

   We will use these later on.

3. Find $Var(X-Y)$.

   - Solution to 3:

$$\begin{aligned}
Var(X-Y) &= \mathbb{E}[(X-Y)^2] - \mathbb{E}[X-Y]^2 \\
&= \int_0^1 \int_0^1 (x-y)^2 \cdot 4(x-xy)dydx - \frac{1}{9} \\
&= \int_0^1 \int_0^1 (x^2-2xy+y^2) \cdot 4(x-xy)dydx - \frac{1}{9} \\
&= 4\int_0^1 \int_0^1 (x^3 - x^3y - 2x^2y + 2x^2y^2 + xy^2 - xy^3)dydx - \frac{1}{9} \\
&= 4\int_0^1 (x^3y - \frac{1}{2}x^3y^2 - x^2y^2 + \frac{2}{3}x^2y^3 + \frac{1}{3}xy^3 - \frac{1}{4}xy^4)_0^1 dx - \frac{1}{9} \\
&= 4\int_0^1 (x^3 - \frac{1}{2}x^3 - x^2 + \frac{2}{3}x^2 + \frac{1}{3}x - \frac{1}{4}x)dx - \frac{1}{9} \\
&= 4(\frac{1}{4}x^4 - \frac{1}{8}x^4 - \frac{1}{3}x^3 + \frac{2}{9}x^3 + \frac{1}{6}x^2 - \frac{1}{8}x^2)_0^1 - \frac{1}{9} \\
&= 4(\frac{1}{4} - \frac{1}{8} - \frac{1}{3} + \frac{2}{9} + \frac{1}{6} - \frac{1}{8}) - \frac{1}{9} \\
Var(X-Y) &= \frac{1}{9}
\end{aligned}$$

   Again, if we look carefully at our algebra, we will see that we have computed $\mathbb{E}[X^2]$, $\mathbb{E}[Y^2]$, and $\mathbb{E}[XY]$:

$$\begin{aligned}
\mathbb{E}[X^2] &= \frac{1}{2} \\
\mathbb{E}[Y^2] &= \frac{1}{6} \\
\mathbb{E}[XY] &= \frac{2}{9}
\end{aligned}$$

4. What is the value of the correlation coefficient, $\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$, of X and Y?

   - Solution to 4: We need to compute all three pieces of the correlation cofficient. We start with the covariance, using the moments obtained above:

$$\begin{aligned}
Cov(X,Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\
&= \frac{2}{9} - \frac{2}{3} \cdot \frac{1}{3} \\
Cov(X,Y) &= 0
\end{aligned}$$

$$\begin{aligned}
Var(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \frac{1}{2} - \left(\frac{2}{3}\right)^2 \\
Var(X) &= \frac{1}{18}
\end{aligned}$$

$$\begin{aligned} Var(Y) &= \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \\ &= \frac{1}{6} - \left(\frac{1}{3}\right)^2 \\ Var(Y) &= \frac{1}{18} \end{aligned}$$

Now, it is certainly clear that

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{0}{\sqrt{\frac{1}{18} \cdot \frac{1}{18}}} = 0.$$

For those of you who noticed the rectangular support for X and Y as well as the ability to factor the joint PDF, $f(x,y) = 4(x - xy)$ into $f_X(x) = 2x$ and $f_Y(y) = 2(1-y)$, you would've seen right away that X and Y were independent, meaning that the correlation coefficient should be zero since the covariance would be zero for two independent random variables.

5. What is $\mathbb{E}[Y|x]$?

- Solution to 5: In order to get $\mathbb{E}[Y|x]$ we need to compute the conditional density using the marginal of X which we guessed above upon recognizing the zero covariance for a linear density (zero correlation does not imply independence–consider a uniform density over $[-1,1]$ for X and $Y = X^2$ which are clearly not independent, but their covariance will actually be zero):

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{4x(1-y)}{2x} = 2(1-y).$$

But, this is obvious, since X and Y are independent, $f(y|x) = f_Y(y)$ which means that $\mathbb{E}[Y|x] = \mathbb{E}[Y] = \frac{1}{3}$.

Question Six

(Bain/Engelhardt p. 191)

Let X and Y have joint pdf $f(x,y) = e^{-y}$ if $0 < x < y < \infty$ and zero otherwise. Find $\mathbb{E}[X|y]$.

- Solution: We've already seen this style of problem before. What we need to do is obtain the distribution of X conditional on $Y = y$ so we can then compute its expectation. We first need the marginal distribution of Y in order to do this. We compute it:

$$\begin{aligned} f_Y(y) &= \int_0^y e^{-y} dx \\ &= xe^{-y}|_0^y \\ f_Y(y) &= ye^{-y} \end{aligned}$$

and then plug it into the formula for the conditional distribution of $X|y$:

$$f(X|y) = \frac{f(x,y)}{f_Y(y)} = \frac{e^{-y}}{ye^{-y}} = \frac{1}{y}.$$

With this, we can now compute the conditional expectation:

$$\begin{aligned}\mathbb{E}[X|y] &= \int_0^y \frac{1}{y} x dx \\ &= \frac{1}{y}\frac{1}{2}x^2|_{x=0}^{x=y} \\ \mathbb{E}[X|y] &= \frac{y}{2}.\end{aligned}$$

Question Seven

Let X be a uniform random variable defined over the interval (a,b), i.e. $f(x) = \frac{1}{b-a}$. The k^{th} central moment of X is defined as $\mu_k = \mathbb{E}[(X-\mathbb{E}[X])^k]$. The standardized central moment is defined as $\frac{\mu_k}{(\mu_2)^{\frac{k}{2}}}$. Find an expression for the k^{th} standardized central moment of X.

- Solution: We just need to figure out an expression for the second central moment (the variance) and then generalize the formula to the k^{th} moment and then plug in the components to the formula. First, note that for the uniform distribution, its expectation is just the midpoint between the endpoints of the support: $\mathbb{E}[X] = \frac{b+a}{2}$.

$$\mathbb{E}[(X-\mathbb{E}[X])^k] = \int_a^b \frac{1}{b-a}\left(x - \frac{b+a}{2}\right)^k dx$$

We can easily solve this for $k=2$:

$$\begin{aligned}\mathbb{E}[(X-\mathbb{E}[X])^2] &= \int_a^b \frac{1}{b-a}\left(x-\frac{b+a}{2}\right)^2 dx \\ &= \frac{1}{b-a}\int_a^b \left(x^2 - (b+a) + \frac{(b+a)^2}{4}\right) dx \\ &= \frac{1}{b-a}\left(\frac{1}{3}x^3 - \frac{(b+a)}{2}x^2 + \frac{(b+a)^2}{4}x\right)_a^b \\ &= \frac{1}{b-a}\left(\frac{1}{3}(b^3-a^3) - \frac{(b+a)}{2}(b^2-a^2) + \frac{(b+a)^2}{4}(b-a)\right) \\ &= \frac{1}{b-a}\left(\frac{1}{3}(b-a)(b^2+ab+a^2) - \frac{(b+a)^2}{4}(b-a)\right) \\ &= \frac{1}{3}b^2 + \frac{1}{3}ab + \frac{1}{3}a^2 - \frac{1}{4}b^2 - \frac{1}{4}2ab - \frac{1}{4}a^2 \\ &= \frac{1}{12}b^2 - \frac{1}{6}ab + \frac{1}{12}a^2 \\ \mathbb{E}[(X-\mathbb{E}[X])^2] &= \frac{1}{12}(b-a)^2\end{aligned}$$

The observant reader will notice that we went to a lot of trouble to expand and then factor the expressions above. What we will now do is a change of variables in order to get the k^{th} moment:

$$\begin{aligned}
\mathbb{E}[(X-\mathbb{E}[X])^k] &= \int_a^b \frac{1}{b-a}\left(x-\frac{b+a}{2}\right)^k dx \\
&= \frac{1}{b-a}\int_{\frac{a-b}{2}}^{\frac{b-a}{2}} z^k dz \\
&= \frac{1}{b-a}\frac{1}{k+1} z^{k+1}\Big|_{\frac{a-b}{2}}^{\frac{b-a}{2}} \\
&= \frac{1}{b-a}\frac{1}{k+1}\left[\left(\frac{b-a}{2}\right)^{k+1} - \left(\frac{a-b}{2}\right)^{k+1}\right] \\
&= \frac{1}{2^{k+1}(k+1)}\left[(b-a)^k + (a-b)^k\right] \\
\mathbb{E}[(X-\mathbb{E}[X])^k] &= \left(1+(-1)^k\right)\frac{(b-a)^k}{2^{k+1}(k+1)}
\end{aligned}$$

That was substantially easier than obtaining the second moment. A quick check of our formula for $k=2$ ensures that we got it right: $\frac{2(b-a)^2}{2^3 \cdot 3} = \frac{(b-a)^2}{12}$. We now apply the formula below:

$$\begin{aligned}
\frac{\mu_k}{(\mu_2)^{\frac{k}{2}}} &= \frac{\mathbb{E}[(X-\mathbb{E}[X])^k]}{\mathbb{E}[(X-\mathbb{E}[X])^2]^{\frac{k}{2}}} \\
&= \frac{\left(1+(-1)^k\right)\frac{(b-a)^k}{2^{k+1}(k+1)}}{\left(\frac{(b-a)^2}{12}\right)^{\frac{k}{2}}} \\
&= \frac{\left(1+(-1)^k\right)\frac{(b-a)^k}{2^{k+1}(k+1)}}{\frac{(b-a)^k}{2^k\left(\sqrt{3}\right)^k}} \\
\frac{\mu_k}{(\mu_2)^{\frac{k}{2}}} &= \left(1+(-1)^k\right)\frac{\left(\sqrt{3}\right)^k}{2(k+1)}
\end{aligned}$$

Now we have a formula for the k^{th} standardized central moment of the uniform distribution. Generate some uniformly distributed random numbers and check the formula!