14.23 Spring 2003
Problem Set 2
Model Solutions

1. A natural monopoly

   (a) (10 points) Marginal costs are $MC = 25$ at regardless of the level of output so

   $$Q_{MC} = 200 - 2 \times 25 = 150. \tag{1}$$

   With constant marginal cost, the revenue PQ is just equal to variable cost $25Q$. Monopolist is only left with the fixed cost 400.

   $$\Pi = -400 \tag{2}$$

   Using the inverse demand function $P = 100 - \frac{Q}{2}$ and the usual triangle method, consumer surplus is

   $$CS = (100 - 25)\,150 \times \frac{1}{2} = 5625. \tag{3}$$

   This is not feasible because the monopolist is making a loss. If the regulator really set the price at marginal cost, then the monopolist would rather go out of business.

   (b) (10p) Setting price equal to average cost $C(Q)/Q$, the equilibrium condition is

   $$\begin{aligned} 100 - \frac{Q}{2} &= \frac{400}{Q} + 25 \\ \Longrightarrow 400 - 75Q + \frac{1}{2}Q^2 &= 0 \end{aligned} \tag{4}$$

   This quadratic equation has two solutions: $Q = 75 \pm \sqrt{193}$. Of these the higher one (144.5) makes more sense than the other (5.54) because it leads to higher consumer surplus. Profits are zero by definition when P=AC, so calculating the profits can only cause errors. So $Q_{AC} = 144.5$, $P = 100 - \frac{144.5}{2} = 27.8$, and

   $$CS = (100 - 27.8)\,144.5 \times \frac{1}{2} = 5217.3. \tag{5}$$

   (c) (10p) With price equal to marginal cost, consumer surplus (before participation fee) would be 562.5 for each of the ten consumers. Knowing this the monopolist can set the participation fee at 562.5, which is the highest price that consumers will pay knowing that they will be charged $P = 25$ once in and will then buy $\frac{Q_{MC}}{10} = 15$. Counting the participation fee, total consumer surplus is zero.

The revenue from marginal cost pricing just exactly offsets the variable cost of production (because of constant marginal cost). Monopolists' profits are therefore equal to participation fees less fixed cost:

$$\Pi = 10 \times 562.5 - 400 = 5225. \tag{6}$$

Total surplus is maximized, if the monopolist is allowed to freely set two-part pricing, but consumers get none of it. The monopolist sets the linear price P as if to maximize consumer surplus, so $P = MC$, and then takes it all away with participation fees.

2. Two-product monopolist.

   (a) (10p) Inverse demands are

$$\begin{aligned} P_1 &= 2 - Q_1 & (7) \\ P_2 &= 4 - 4Q_2 & (8) \end{aligned}$$

   Consider a general linear inverse demand $P_i = \hat{P}_i - \alpha_i Q_i$, where $\hat{P}_i$ is the choke price (defined as the price above which demand is zero). The demand is then $Q_i = \frac{1}{\alpha_i}\left(\hat{P}_i - P_i\right)$ and the consumer surplus, as a function of the price, is

$$\begin{aligned} &\left(\hat{P}_i - P_i\right) Q_i \frac{1}{2} = \\ &\left(\hat{P}_i - P_i\right)\left(\hat{P}_i - P_i\right)\frac{1}{\alpha_i}\frac{1}{2} = \frac{1}{2\alpha_i}\left(\hat{P}_i - P_i\right)^2 \end{aligned} \tag{9}$$

   Plugging in the parameters $\{\hat{P}_1 = 2, \hat{P}_2 = 4, \alpha_1 = 1, \alpha_2 = 4\}$ yields

$$\begin{aligned} CS(P_1, P_2) &= \frac{1}{2}(2 - P_1)^2 + \frac{1}{8}(4 - P_2)^2 & (10) \\ &= -2P_1 + \frac{1}{2}P_1^2 - P_2 + \frac{1}{8}P_2^2 + 4 & (11) \end{aligned}$$

   This is only sensible for $P_1 \in [0, 2], P_2 \in [0, 4]$.

   (b) (10p) According to the textbook (page 353), the demand at Ramsey prices satisfies

$$\frac{Q_{MC,1} - Q_1}{Q_{MC,1}} = \frac{Q_{MC,2} - Q_2}{Q_{MC,2}} \tag{12}$$

   where $Q_{MC,i}$ is the demand when price equals marginal cost. (This result requires linear demand functions!) In other words, the output of both products are reduced in same proportion compared to the levels of output that would be demanded at marginal cost prices. Here marginal costs are zero, so this becomes

$$\frac{2-Q_1}{2} = \frac{1-Q_2}{1} \Longrightarrow Q_1 = 2Q_2. \tag{13}$$

In terms of prices this is

$$\begin{aligned} 2 - P_1 &= 2\left(1 - \frac{1}{4}P_2\right) && (14) \\ &\Rightarrow P_2 = 2P_1 && (15) \end{aligned}$$

We also know that revenue must be equal to costs, which are fixed at 1 (in millions \$), so the other condition that must hold is

$$P_1\left(2 - P_1\right) + P_2\left(1 - \frac{1}{4}P_2\right) = 1. \tag{16}$$

Substituting in (15) and solving the resulting quadratic equation $2P_1^2 - 4P_1 + 1 = 0$ yields $P_1 = 1 \pm \frac{1}{\sqrt{2}}$, of which the lower price is the more sensible solution. (Again, consumer surplus is higher in the more sensible solution, while the monopolist makes zero profits in both).

$$\begin{aligned} P_1^R &= 1 - \frac{1}{\sqrt{2}} = 0.293 && (17) \\ P_2^R &= 2\left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2} = 0.586 && (18) \\ Q_1^R &= 2 - P_1 = 1 + \frac{1}{\sqrt{2}} = 1.707 && (19) \\ Q_2^R &= 1 - \frac{1}{4}P_2 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) = 0.854 && (20) \end{aligned}$$

Finally, plugging in the above solution into consumer surplus equation (10) results in

$$\begin{aligned} CS^R &= \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{8}\left(\frac{3}{2} - \frac{1}{2\sqrt{2}}\right)^2 \\ &= \cdots = \frac{3}{2} + \sqrt{2} = 2.914. && (21) \end{aligned}$$

(c) (10p) The regulator requires that half of the cost must be paid by consumers of each product. This means that the revenue from each product must be equal to 0.5 \$million:

$$\begin{aligned} P_1Q_1 &= P_2Q_2 = 0.5 && (22) \\ &\Longrightarrow \begin{array}{ll} P_1\left(2 - P_1\right) & = 0.5 \\ P_2\left(1 - \frac{1}{4}P_2\right) & = 0.5 \end{array} && (23) \end{aligned}$$

These two equations give the same (sensible) solutions as the Ramsey prices in part b). It happens that in this case the Ramsey prices resulted in equal total revenue from the two products, so this regulation yields the same prices (and therefore same outputs and welfare) as if the regulator required Ramsey pricing. In general this would not have to be the case.

3. Peak-load pricing.

   This question relates to the textbook pages 379-386. It gives two cases (*the firm peak* case and *the shifting peak* case). The firm-peak case is relevant here, because the off-peak demand is low enough to not affect the optimal capacity.

   (a) (10p) In the firm peak case, the peak users have to pay for all capacity costs, so their price is P^P =LRMC= $0.2 + 0.3 = 0.5$, and peak demand is $Q^P = 10 - 0.5 = 9.5$. Capacity is always equal to peak demand (anything more would be wasted, anything less could not satisfy peak demand): $K = 9.5$.

   Off-peak users will only pay for the variable costs, so P^o =SRMC= 0.2 and $Q^o = 4 - 0.2 = 3.8$. There will be idle capacity in off-peak periods. (If there were not then this would be the shifting peak case after all, and capacity costs would have to be shared between peak and off-peak users).

   (b) (10p) Consumer Surplus and Producer Surplus as a function of the uniform price:

$$\mathrm{CS}(P) = \frac{1}{2}(4-P)^2 + \frac{1}{2}(10-P)^2 \tag{24}$$

$$\mathrm{PS}(P) = \underbrace{P(4-P) + P(10-P)}_{\text{Revenue}} - \underbrace{0.2(14-2P) - 0.3(10-P)}_{\text{Cost}} \tag{25}$$

$$= (P-0.2)(14-2P) - 0.3(10-P) \tag{26}$$

   Total surplus is their sum, which simplifies into a quardatic

$$\mathrm{TS}(P) = \cdots = -P^2 + \frac{7}{10}P + \frac{261}{5}. \tag{27}$$

   This is maximized at

$$-2P + \frac{7}{10} = 0 \Longrightarrow P^* = \frac{7}{20} = 0.35. \tag{28}$$

   Thus peak demand and capacity are $Q^P = 10 - 0.35 = 9.65$, while the off-peak demand is $Q^o = 3.65$.

   (c) (10p) With peak-load pricing, the consumer social surplus is

$$(3.8)^2 \frac{1}{2} + (9.5)^2 \frac{1}{2} = 52.35. \tag{29}$$

Since revenue is equal to total costs this is also the social surplus. With a uniform price, consumer surplus is

$$(4-0.35)^2 \frac{1}{2} + (10-0.35)^2 \frac{1}{2} = 53.223. \tag{30}$$

Consumer surplus is higher with uniform pricing only because producer producer surplus is negative. It is PS(P)=−0.778, so social surplus is 52.2, lower than with peak-load pricing (there is too much capacity). Therefore social surplus is lower when pricing is constrained to be uniform across periods, even if the producer were allowed to make a loss.

4. The technology does not cause there to be a natural monopoly if it is not strictly cheaper to produce output (x, y) in one firm than in two firms. There is no natural monopoly if for some $a \in (0, x), b \in [0, y]$

$$C(x,y) \geq \min \{C(x-a, y-b) + C(a,b)\}. \tag{31}$$

   (a) (3p) Linear costs are the same no matter how many firms output is divided between—not a natural monopoly.

   (b) (3p) Costs are zero if two firms specialize in one good:

$$C(x,y) > C(x,0) + C(0,y) \tag{32}$$

   There are diseconomies of scope, so this is not a natural monopoly.

   (c) (4p) This is a more tricky case, because the result depends on (x, y). First note that the cheapest way to produce any output in two firms is to make both firms specialize: $C(x,0) = C(0,y) = 1$. Minimized total cost of two-firm production is therefore 2 regardless of the levels (x, y). A monopolist producing some of each product has to pay the fixed cost only once, but also has to pay variable costs $\sqrt{xy}$. There are diseconomies of scope, but also increasing returns to scale due to the fixed cost. Monopolists costs are lower if

$$\sqrt{xy} + 1 \quad < \quad 2 \tag{33}$$

$$\Rightarrow \quad x < \frac{1}{y}. \tag{34}$$

   This is the condition for a natural monopoly. If the demand for one of the products is much lower than for the other, than the diseconomies of scope do not matter very much and there is a natural monopoly. However, if there is sufficient demand for both products then it is more efficient to have two specialized firms.