Economics of Networks
Diffusion Part 2

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Agenda

- Recap of last time, contagion and mean-field diffusion
- The configuration model
- Diffusion in random graphs
- Monopoly pricing with word-of-mouth communication

Material not well-covered in one place. Some suggested reading: Jackson Chapter 7.2; "Word-of-Mouth Communication and Percolation in Social Networks," A. Campbell; "Diffusion Games," E. Sadler

Binary Coordination with Local Interactions

Recall our simple game:

	0	1
0	(q, q)	$(0, 0)$
1	$(0, 0)$	$(1-q, 1-q)$

Two pure-strategy equilibria

Play simultaneously with many neighbors

- Choose 1 if at least fraction q of neighbors choose 1
- Myopic best response, can the action spread?

Cohesion can block contagion, but neighborhoods can't grow too fast

Mean-Field Diffusion

An alternative framework

- Distributional knowledge of the network structure
- Adopt behavior iff expected payoff exceeds cost

Bayes-Nash equilibrium of static game equivalent to steady-state of mean-field dynamics

- Draw a new set of neighbors a every time step

Key phenomenon: tipping points

Relate steady state to network degree distribution

Random Graphs

Today, a third approach

- Distributional knowledge of the network structure
- Diffusion through a fixed graph

People get exposed to something new

- Behavior, product, information...

Choose whether to adopt or pass it on

Stochastic outcomes, viral cascades

A Few Examples

Spread of new products through referral programs

Spread of rumors about Indian demonetization policy (Banerjee et al., 2017)

Spread of news stories on social media

- Maybe fake ones...

Spread of microfinance participation (Banerjee et al., 2013)

Questions

Basic:

- How many people adopt?
- How quickly does it spread?
- Who ends up adopting?

Some implications:

- Targeted seeding
- Pricing strategies

First: The Configuration Model

Recall the configuration model from the first half of the course

- This is how we will generate our networks

Start with n nodes, degree sequence $\mathbf{d}^{(n)} = (d_1, d_2, ..., d_n)$

- Degree is number of neighbors a node has

Take a uniform random draw of graphs with the given degree sequence

Look at limits as $n \to \infty$, large networks

- Assume $\{\mathbf{d}^{(n)}\}$ converges in distribution and expectation to D

The Configuration Model

Can think of D as a histogram

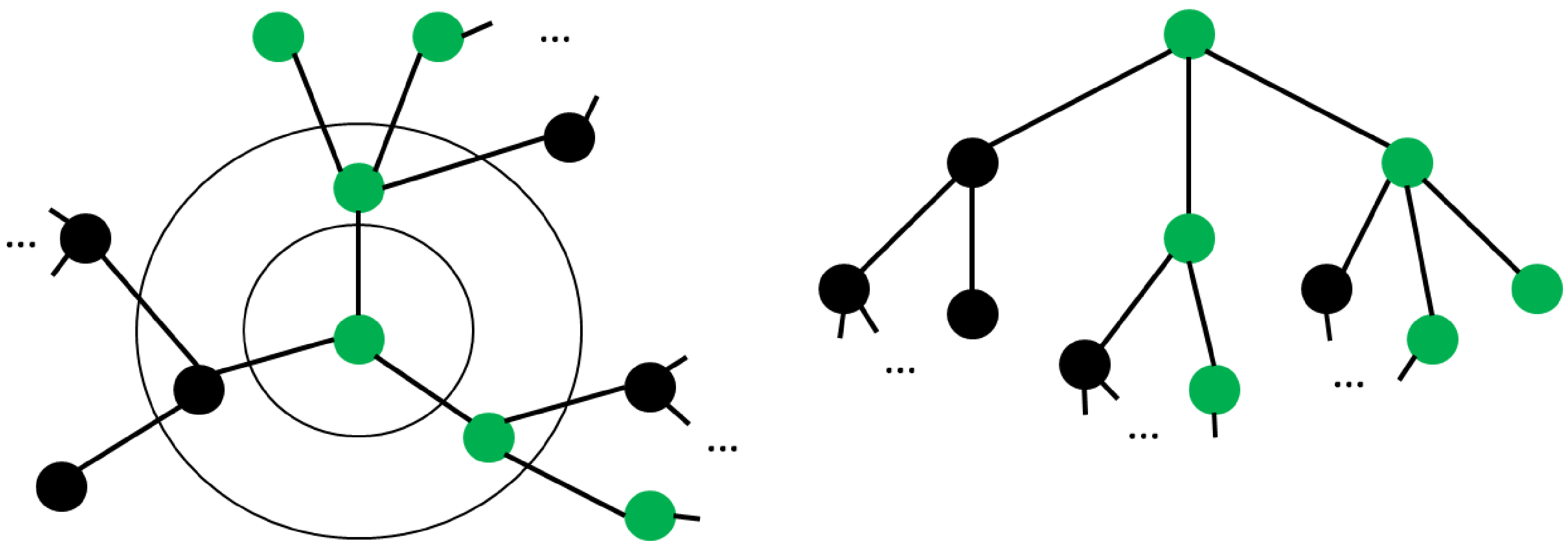
- $\mathbb{P}(D = k)$ is the fraction of nodes with degree k

Questions about the configuration model:

- What do the components look like?
- Is there a "giant" component?
- How big is it?
- How far are typical nodes from each other?

Key idea: branching process approximation

Branching Process Approximation



Branching Processes

Let $Z \in \Delta(\mathbb{N})$ be a probability distribution

Start from a single root note, realize offspring according to Z

Each node in the first generation realizes offspring independently according to Z

And so on...

Branching Processes: Extinction

What is the probability that the process goes extinct?

- Total number of offspring is finite

Key tool: the generating function

$$g(s) = \sum_{k=0}^{\infty} \mathbb{P}(Z = k) s^k$$

Well-defined for $k \in [0, 1]$

Solve recursion: probability I go extinct is probability all my offspring go extinct

$$\xi = \sum_{k=0}^{\infty} \mathbb{P}(Z = k) \xi^k = g(\xi)$$

Extinction probability is unique minimal solution to $\xi = g(\xi)$

- Survival probability $\phi = 1 - \xi$

Branching Processes: Growth Rate

Expected number of offspring $\mathbb{E}[Z] \equiv \mu$

- Generation t contains μ^t nodes in expectation

Write Z_t for the random number of total offspring in generation t

- The process $Y_t \equiv \frac{Z_t}{\mu^t}$ is a martingale

By the martingale convergence theorem:

- As $t \to \infty$, Y_t converges almost surely
- Implication: $\phi > 0$ i $\mu > 1$ (one exception: $Z = 1$ w.p.1)

Connecting to the Random Graph

Heuristically, breadth first search starting from a random node looks like a branching process

- The "characteristic branching process" $\mathcal{T}$ for the graph

Root realizes offspring according to D

After the root, two corrections

- Friendship paradox
- Don't double count the parent

Subsequent nodes realize offspring according to D'

$$\mathbb{P}(D' = d) = \frac{\mathbb{P}(D = d+1) \cdot (d+1)}{\mathbb{E}[D]}$$

The Law of Large Networks

Define $\rho_k = \mathbb{P}(|\mathcal{T}| = k)$, $N_k(G)$ the number of nodes in components of size k, $L_i(G)$ the ith largest component

Theorem

Suppose $\mathbf{d}^{(n)} \to D$ in distribution and expectation, and $G^{(n)}$ is generated from the configuration model with degree sequence $\mathbf{d}^{(n)}$. For any $\epsilon > 0$, we have

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{N_k(G^{(n)})}{n} - \rho_k\right| > \epsilon\right) = 0, \quad \forall\, k$$

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{L_1(G^{(n)})}{n} - \rho_\infty\right| > \epsilon\right) = 0$$

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{L_2(G^{(n)})}{n} > \epsilon\right) = 0.$$

The Law of Large Networks

For large graphs, network structure completely characterized by the branching process

- Distribution of component sizes
- Size of giant component

Note: need D' non-singular

Proof is beyond our scope

Survival Probability of $\mathcal{T}$

Fun fact: if $g(s)$ is the generating function for D, then $\frac{g'(s)}{\mu}$ is the generating function for D':

$$g'(s) = \frac{\mathrm{d}}{\mathrm{d}s} g(s) = \frac{\mathrm{d}}{\mathrm{d}s} \sum_{k=0}^{\infty} \mathbb{P}(D = k) s^k$$
$$= \sum_{k=1}^{\infty} k \mathbb{P}(D = k) s^{k-1}$$
$$= \sum_{k=0}^{\infty} (k+1) \mathbb{P}(D = k+1) s^k$$

If ξ solves $\mu\xi = g'(\xi)$, survival probability of $\mathcal{T}$ is $\phi = 1 - g(\xi)$

- Giant component covers fraction ϕ of the network

Typical Distances

Define $\nu = \mathbb{E}[D']$, $H(G)$ distance between two random nodes in the largest component of G

Theorem

A giant component exists if and only if $\nu > 1$. In this case, for any $\epsilon > 0$ we have

$$\lim_{n \to \infty} \mathbb{P}\left(\left| \frac{H(G)}{\log_\nu n} - 1 \right| > \epsilon \right) = 0$$

Typical distance between nodes is $\log_\nu n$

- Relates to growth rate of the branching process $\mathcal{T}$

A Diffusion Process

People learn about a product through word-of-mouth

- i.i.d private values v distributed uniformly on $[0, 1]$
- Price p

If I learn about the product, buy if $v > p$

- If I buy, my friends hear about it, make their own choices

Suppose n individuals are linked in a configuration model, and one random person starts out with the product

- How many people end up buying?
- How long does it take to spread?

Outcome Variables

Let $X_n(t)$ denote the (random) number of purchasers after t periods in the n person network

Define extent of adoption

$$\alpha_n = \lim_{t \to \infty} \frac{X_n(t)}{n}$$

For $x \in (0, 1)$, diffusion times

$$\tau_n(x) = \min \left\{ t : \frac{X_n(t)}{X_n(\infty)} \geq x \right\}$$

Will characterize α_n and τ_n for large n

Percolation in the Configuration Model

If I buy, each neighbor will buy with independent probability $1 - p$

- Adoption spreads through a subgraph
- As if we delete each person with independent probability p

The percolated graph is also a configuration model, degree distribution D_p

- Realize degree according to D, delete each link with probability p
- Binomial distribution with D trials and success probability $1 - p$

Generating function for D_p:

$$g_p(s) = g\left(p + (1-p)s\right)$$

The Extent of Diffusion

Recall $\mu = \mathbb{E}[D]$

Theorem

There exist ϕ_p and ζ_p such that α_n converges in distribution to a random variable α, taking the value ϕ_p with probability ζ_p and the value 0 otherwise. To obtain these constants, we can solve

$$\mu\xi = g'\left(p + (1-p)\xi\right)$$

If ξ^* is the solution, we have $\phi_p = (1-p)\left(1 - g_p(\xi^*)\right)$ and $\zeta_p = 1 - g_p(\xi^*)$.

Intuitively, question is whether the initial seed touches the giant component in the percolated graph

- ϕ_p is fraction of nodes in the component, ζ_p is fraction of nodes that link to this component

The Extent of Diffusion

$\mathbb{E}[D_p] = (1-p)\mu$, $g_p'(s) = (1-p)g'(p + (1-p)s)$

- ξ^* is the extinction probability of a non-root node
- By the law of large networks, this is the probability that, going forward, a node does not link to the giant component

Probability that I do not link to the giant component:

$$\sum_{k=1}^{\infty} \mathbb{P}(D_p = k)(\xi^*)^k = g_p(\xi^*)$$

Probability that I purchase: $1-p$

The Rate of Diffusion

Theorem

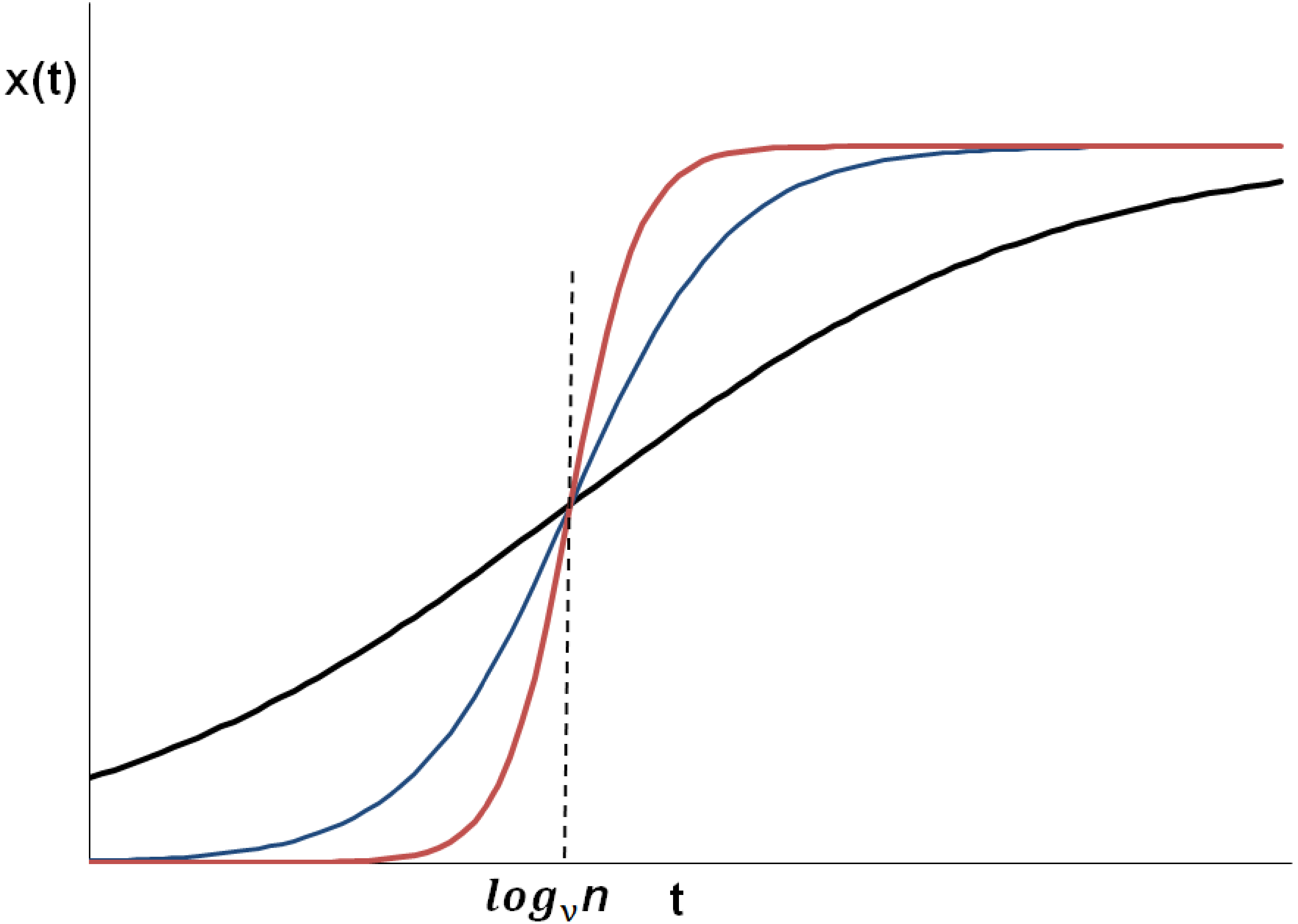
Conditional on having a large cascade, for all $x \in (0,1)$ we have

$$\frac{\tau_n(x)}{\log_{p\nu} n} \to 1$$

in probability.

The time it takes to reach any positive fraction is roughly $\log_{p\nu} n$

The Rate of Diffusion



Comparative Statics

How does adoption change with the price and the network?

Theorem

Suppose $\nu > 1$. There exists a critical price $p_c \in (0,1)$ such that

- $\phi_p = \zeta_p = 0$ *for* $p \geq p_c$
- $\phi_p > 0$ *for* $p < p_c$*, and* $\frac{\partial \phi_p}{\partial p} < 0$

Suppose D and $\hat{D}$ are two distributions, withϕ_p and $\hat{\phi}_p$ the corresponding giant component sizes, and ν and $\hat{\nu}$ the corresponding forward degrees. If D FOSD $\hat{D}$, then $\phi_p \geq \hat{\phi}_p$ and $\nu \geq \hat{\nu}$. If D is a mean preserving spread of $\hat{D}$, then $\nu \geq \hat{\nu}$

Comparative Statics

For suffciently high prices, there is no adoption, impossible to get viral cascade

Below the critical price, adoption is decreasing in price

- At p_c, derivative makes a discontinuous jump

Making the network more dense leads to more adoption and faster diffusion

- Mean preserving spread makes diffusion faster, but may not lead to more adoption

Example

Suppose D takes the value 3 for sure, $\hat{D}$ takes values 1 or 5 with equal probability

Under D, extinction probability solves

$$\xi = (p + (1-p)\xi)^2 \quad \Longrightarrow \quad \xi = \min\left\{1, \left(\frac{p}{1-p}\right)^2\right\}$$

For p close to zero, ξ is close to zero, $\phi_p \approx 1$

Under $\hat{D}$, extinction probability solves

$$6\xi = 1 + 5\left(p + (1-p)\xi\right)^4$$

For p close to zero, ξ close to 0.17, $\phi_p \approx 0.83$

A Pricing Problem

Suppose a monopolist is selling this product and wants to choose p to maximize profits

- Constant marginal cost $c < 1$

Assume a fraction $\epsilon \approx 0$ of the population gets seeded at random

- For large networks, guaranteed to hit the giant component

Total demand is fraction $Q(p) = \phi_p = (1 - p)\,(1 - g_p(\xi^*))$ of the population

Choose p to maximize $Q(p)(p - c)$

A Pricing Problem

If all consumers were exposed to the product, then $Q(p) = 1 - p$

- Maxmizie $(1-p)(p-c)$
- Set $p = \frac{1+c}{2}$, profit $\frac{(1-c)^2}{4}$
- Price elasticity: $\frac{p}{Q(p)}\frac{\partial Q}{\partial p} = -\frac{p}{1-p}$

With word of mouth, $Q(p) = (1-p)\left(1 - g_p(\xi^*)\right)$, strictly less

- Demand is also more elastic:

$$\frac{p}{Q(p)}\frac{\partial Q}{\partial p} = -\frac{p}{1-p}\left(1 + \frac{(1-p)(1-\xi^*)g'\left(p+(1-p)\xi^*\right)}{1-g_p(\xi^*)}\right)$$

Implies lower optimal price

Price Comparative Statics

Recall Poisson distribution:

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Mean and variance λ

Theorem (Campbell, 2013)

Suppose the degree distribution is Poisson with parameter λ. The optimal monopoly price is increasing in λ.

Dense network leads to higher prices

- Intuition: monopolist less reliant on any individual spreading information

Advertising

Suppose now our monopolist can invest in advertising in addition to word-of-mouth

Can inform a fraction ω of the population at cost $\alpha\omega$ for $\alpha > 0$

New objective, maximize

$$Q(p, \omega)(p - c) - \alpha\omega$$

Quantity depends now both on price and on advertising ω

Advertising

Word-of-mouth complements advertising

- Customers exposed through advertising will inform additional people

Hard to jointly solve for optimal p and ω, but...

Theorem

Suppose the degree distribution is Poisson with parameter λ. Price and advertising are strategic complements.

All else equal, higher prices tend to go with more advertising

Takeaways

Discrete network diffusion models help us think about viral cascades

- Component sizes in percolation network

Faster diffusion $\neq$ more diffusion

Word-of-mouth leads to more elastic demand, tends to lower prices

Next time: models of network formation