Economics of Networks
Network Effects: Part 2

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Agenda

Local network effects

No textbook covers this material yet, three good papers:

- Bramoullé, Kranton, and D'Amours (2014), "Strategic Interaction and Networks," American Economic Review
- Ballester, Calvó-Armengol, and Zenou (2006), "Who's Who in Networks. Wanted: The Key Player," Econometrica
- Candogan, Bimpikis, and Ozdaglar (2012), "Optimal Pricing in Networks with Externalities," Operations Research

Local Network Effects

So far, focus on homogeneous externalities

Spillovers often depend on individual identities and relationships

- Searching for job opportunities
- Academic peer effects
- Learning spillovers
- Crime
- Oligopoly

Can study network games to gain insight into how relationship patterns affect effort incentives

General Framework

Set of players $N = \{1, 2, ..., n\}$

Each player chooses an action $x_i \geq 0$

- Action profile $x = (x_1, x_2, ..., x_n)$

Players in an undirected interaction network

- Adjacency matrix G with entries $g_{ij} \in \{0, 1\}$

Player i's payoff $U_i(x_i, x_{-i}, \delta, G)$

- Parameter $\delta \geq 0$ captures role of interactions

Strategic Substitutes

Define the payoffs as

$$U_i(x_i, x_{-i}, \delta, G) = b_i\left(x_i + \delta \sum_{j \neq i} g_{ij} x_j\right) - k_i x_i$$

where b_i is differentiable, strictly increasing, and concave in x_i

- Assume $b_i'(\infty) < k_i < b_i'(0)$
- Strategic substitutes

First order condition:

$$b_i'\left(x_i + \delta \sum_{j \neq i} g_{ij} x_j\right) - k_i \leq 0$$

Write $\overline{x}_i$ for solution to $b_i'(x) = k_i$

Best reply is $x_i = \max\left\{0, \overline{x}_i - \delta \sum_{j \neq i} g_{ij} x_j\right\}$

Example: A Cournot Game

Set of N firms produce heterogeneous goods

- Edge between two firms indicates products are substitutes
- Parameter δ indicates degree of substitutability

Firm i faces inverse demand

$$p_i(\mathbf{q}) = a - \left(q_i + \delta \sum_{j \neq i} g_{ij} q_j \right)$$

where $a > 0$

If marginal cost is c, profit is

$$U_i(\mathbf{q}, \delta, G) = q_i \left(a - \left(q_i + \delta \sum_{j \neq i} g_{ij} q_j \right) \right) - c q_i$$

Example: A Cournot Game

First order condition:

$$\frac{\partial U_i}{\partial q_i} = a - \left(q_i + \delta \sum_{j \neq i} g_{ij} q_j \right) - q_i - c = 0,$$

implying

$$q_i = \frac{a - c - \delta \sum_{j \neq i} g_{ij} q_j}{2}$$

Note: we recover the classic model by taking $\delta = g_{ij} = 1$ for all j

Strategic Substitutes

If $x_i > 0$, say i is active, else inactive

For simplicitly, assume function is such that $\overline{x}_i = 1$

- $x_i = \max\left(0, 1 - \delta \sum_{j \neq i} g_{ij} x_j\right)$
- Brouwer's fixed point theorem guarantees equilibrium existence

- Set of active agents A
- Active agent action profile $\mathbf{x}_A$
- Links between active agents G_A
- Links connecting active agents to inactive ones $G_{N-A,A}$

Equilibrium Structure

Proposition

In any Nash equilibrium, the action profile of active agents $\mathbf{x}_A$ *satisfies:*

$$(I + \delta G_A)\mathbf{x}_A = \mathbf{1}$$

$$\delta G_{N-A,A}\mathbf{x}_A \geq \mathbf{1}$$

First condition ensures active players are best-responding

- Compute equilibrium actions as $\mathbf{x}_A = (I + \delta G_A)^{-1} \cdot \mathbf{1}$
- Follows from first order condition

Second condition ensures inactive players are best-responding

Computing Equilibria

How can we find the equilibria?

- Guess and check

Fix a subset of the players $S \subseteq N$ and compute

$$\mathbf{x}_S = (I + \delta G_S)^{-1}\mathbf{1}$$

Then check whether $\delta G_{N-S,S}\mathbf{x}_S \geq \mathbf{1}$

If yes, then we have found an equilibrium with S as the set of active players

Example: Computing Equilibria

Consider four players in a line graph:

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Suppose all players are active:

$$(I + \delta G)^{-1} = \frac{1}{\delta^4 - 3\delta^2 + 1} \begin{pmatrix} 1 - 2\delta^2 & \delta^3 - \delta & \delta^2 & -\delta^3 \\ \delta^3 - \delta & 1 - \delta^2 & -\delta & \delta^2 \\ \delta^2 & -\delta & 1 - \delta^2 & \delta^3 - \delta \\ -\delta^3 & \delta^2 & \delta^3 - \delta & 1 - 2\delta^2 \end{pmatrix}$$

Example: Computing Equilibria

$$(I+\delta G)^{-1}\mathbf{1} = \frac{1}{\delta^4 - 3\delta^2 + 1}\begin{pmatrix} 1-\delta-\delta^2 \\ 1-2\delta+\delta^3 \\ 1-2\delta+\delta^3 \\ 1-\delta-\delta^2 \end{pmatrix} = \frac{1}{1+\delta-\delta^2}\begin{pmatrix} 1 \\ 1-\delta \\ 1-\delta \\ 1 \end{pmatrix}$$

Actions must be non-negative, so we have an equilibrium with all players active if only if $\delta < 1$.

Example: Computing Equilibria

Suppose one of the center players is inactive ($S = \{1, 3, 4\}$)

Only two linked active players (one end is isolated), gives

$$G_S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(I + \delta G_S)^{-1} = \frac{1}{\delta^2 - 1} \begin{pmatrix} -1 & \delta \\ \delta & -1 \end{pmatrix}$$

As long as $\delta \neq 1$, we have

$$(I + \delta G_S)^{-1} \mathbf{1} = \frac{1}{\delta^2 - 1} \begin{pmatrix} \delta - 1 \\ \delta - 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\delta} \\ \frac{1}{1+\delta} \end{pmatrix}$$

Example: Computing Equilibria

The isolated active player 1 chooses $x_1 = 1$, so

$$x_S = \begin{pmatrix} 1 \\ \frac{1}{1+\delta} \\ \frac{1}{1+\delta} \end{pmatrix} \geq 0$$

Need to check for the inactive player 2 that $\delta G_{N-S,S} x_S \geq 1$:

$$\delta G_{N-S,S} x_S = \delta \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{1+\delta} \\ \frac{1}{1+\delta} \end{pmatrix} = \delta \left(1 + \frac{1}{1+\delta} \right)$$

Profile is an equilibrium if $1 > \delta \geq \frac{\sqrt{5}-1}{2}$

The Potential Function

Define the potential

$$\Phi(\mathbf{x}, \delta, G) = \mathbf{x}^T \mathbf{1} - \frac{1}{2}\mathbf{x}^T (I + \delta G)\mathbf{x}$$
$$= \sum_{i=1}^{n} \left(x_i - \frac{1}{2} x_i^2 \right) - \frac{1}{2}\delta \sum_{i,j=1}^{n} g_{ij} x_i x_j .$$

First order conditions for maximizing Φ are same as first order condition for each player's optimization

- Need $1 - x_i - \delta \sum_{j \neq i} g_{ij} x_j \leq 0$

This is a potential game

Uniqueness of Equilibrium

Theorem (Bramoullé et al., 2014)

The set of Nash equilibria given G and δ is the set of local maxima and saddle points of the potential $\Phi(\mathbf{x}, \delta, G)$
If $|\lambda_{min}(G)| < \frac{1}{\delta}$, there is a unique Nash equilibrium.

The KKT conditions for maximizing Φ are exactly the best response conditions for each player

- For each i, we need $0 = 1 - x_i - \delta \sum_{j \neq i} g_{ij} x_j + \mu_i$
- Complementary slackness implies $\mu_i > 0$ only if $x_i = 0$

If Φ is strictly concave, the KKT conditions are necessary and sufficient, so there is a unique solution

Uniqueness Continued

We have $\nabla^2 \Phi = -(I + \delta G)$, so Φ is strictly concave iff $I + \delta G$ is positive definite

$I + \delta G$ is positive definite iff $\lambda_{min}(I + \delta G) > 0$

$\lambda_{min}(I + \delta G) > 0$ iff $\lambda_{min}(G) < \frac{1}{\delta}$

Uniqueness Continued

Proposition

For any graph G, if $|\lambda_{min}(G)| \geq \frac{1}{\delta}$, there exists at least one Nash equilibrium with inactive agents.

In the line graph with four players, we have

$$|\lambda_{min}(G)| = \frac{\sqrt{5}+1}{2} = \frac{2}{\sqrt{5}-1}$$

Recall the equilibrium with an inactive center player required

$$\delta \geq \frac{\sqrt{5}-1}{2} \iff \frac{1}{\delta} \leq \frac{2}{\sqrt{5}-1}$$

Comparative Statics

How do equilibria change when we add links or increase δ? Partial answer...

Theorem

Consider the highest aggregate play equilibrium $x^*(\delta, G)$ for δ and G. Suppose $\delta' \geq \delta$ and $G' \supseteq G$. Then for any equilibrium vector $x(\delta', G')$, we have

$$\sum_{i=1}^{n} x_i(\delta', G') \leq \sum_{i=1}^{n} x_i^*(\delta, G)$$

Adding links or increasing substitutability typically reduces equilibrium play

Strategic Complements

Strategic substitutes capture examples like public goods provision and Cournot competition

In other cases, actions are complements

- Learning spillovers
- Bank runs
- Criminal activity

Suppose payoffs are

$$U_i(x_i, x_{-i}, \delta, G) = x_i - \frac{1}{2}x_i^2 + \delta \sum_{j \neq i} g_{ij} x_i x_j$$

Strategic Complements

First order conditions imply

$$x_i = 1 + \delta \sum_{j \neq i} g_{ij} x_j$$

Theorem

If $\lambda_{max}(G) < \frac{1}{\delta}$, there is a unique Nash equilibrium with actions

$$\mathbf{x} = (I - \delta G)^{-1} \mathbf{1}.$$

The vector $(I - \delta G)^{-1} \mathbf{1} \equiv \mathcal{K}(\delta, G)$ gives the Katz-Bonacich centralities of the players

If $\lambda_{max}(G) > \frac{1}{\delta}$, there is no equilibrium

Key Players

Each player contributes to aggregate activity in proportion to centrality

$$\frac{x_i^*(\delta, G)}{\sum_{j=1}^n x_j^*(\delta, G)} = \frac{\mathcal{K}_i(\delta, G)}{\sum_{j=1}^n \mathcal{K}_j(\delta, G)}$$

Suppose this is a model of criminal activity, and we want to reduce aggregate crime by targeting key individuals

- Who do we target?

Write G^{-i} for the network without player i, solve

$$\min \left\{ \sum_{j \neq i} x_j^*(\delta, G^{-i}) \,|\, i = 1, 2, ..., n \right\}$$

We call the solution i^* the key player

Key Players

Theorem

If $\lambda_{max} < \frac{1}{\delta}$, the key player i^* has the highest intercentrality

$$c_i(\delta, G)\frac{\mathcal{K}_i(\delta, G)^2}{m_{ii}(\delta, G)}$$

where $M(\delta, G) = (I - \delta G)^{-1}$

Intercentrality is different from Katz-Bonacich centrality

Intuitively, need to capture not only a player's activity level (proportional to Katz-Bonacich centrality), but the player's contribution to others' centralities as well

Key Players: Proof

When $M(\delta, G)$ is well defined, we have

$$m_{ji}(\delta, G) m_{ik}(\delta, G) = m_{ii}(\delta, G) \left(m_{jk}(\delta, G) - m_{jk}(\delta, G^{-i}) \right)$$

$$\begin{aligned}
\sum_j \mathcal{K}_j(\delta, G) - \sum_j \mathcal{K}_j(\delta, G^{-i}) \\
&= \mathcal{K}_i(\delta, G) + \sum_{j \neq i} \mathcal{K}_j(\delta, G) - \mathcal{K}_j(\delta, G^{-i}) \\
&= \mathcal{K}_i(\delta, G) + \sum_{j \neq i} \sum_{k=1}^{N} \left(m_{jk}(\delta, G) - m_{jk}(\delta, G^{-i}) \right) \\
&= \mathcal{K}_i(\delta, G) + \sum_{j \neq i} \sum_{k=1}^{N} \frac{m_{ji}(\delta, G) m_{ik}(\delta, G)}{m_{ii}(\delta, G)} \\
&= \frac{\mathcal{K}_i(\delta, G)}{m_{ii}(\delta, G)} \left(m_{ii}(\delta, G) + \sum_{j \neq i} m_{ji}(\delta, G) \right)
\end{aligned}$$

Pricing-Consumption Model

Now suppose we want to price a good that entails local externalities

- How should we set prices?
- How much is information about the network worth?

Set of agents $N = \{1, 2, ..., n\}$, weighted network G

- Interpret g_{ij} as influence of j on i
- Assume $g_{ij} \geq 0$, $g_{ii} = 0$
- Do not need $g_{ij} = g_{ji}$

Monopolist produces a good, chooses vector $\mathbf{p}$ of prices

- Perfect price discrimination: charge p_i to agent i

Pricing-Consumption Model

Agent's utility:

$$u_i(x_i, x_{-i}, p_i) = a_i x_i - b_i x_i^2 + x_i \sum_{j \neq i} g_{ij} x_j - p_i x_i$$

- Direct benefit $a_i x_i - b_i x_i^2$
- Social benefit
- Price

Two stage game

- Monopolist chooses prices $\mathbf{p}$ to maximize $\sum_i p_i x_i - c x_i$
- Agents choose usages x_i to maximize utilities $u_i(\mathbf{x}, p_i)$
- Look at subgame perfect equilibria

Consumption Equilibrium

Work backwards, taking prices as given

Define diagonal matrix Λ with $\Lambda_{ii} = 2b_i$, let $S \subseteq N$ be a subset of the agents

Theorem

Assume $2b_i > \sum_{j \in N} g_{ij}$ for all i. For any $\mathbf{p}$, there is a unique consumption equilibrium of the form

$$\mathbf{x}_S = (\Lambda_S - G_S)^{-1}(\mathbf{a}_S - \mathbf{p}_S)$$

$$\mathbf{x}_{N-S} = \mathbf{0}$$

for some subset $S \subseteq N$

Optimal Pricing

Theorem

Assume $a_i > c$ for all $i \in N$. The optimal prices are given by

$$\mathbf{p} = \mathbf{a} - (\Lambda - G)\left(\Lambda - \frac{G + G^T}{2}\right)^{-1} \frac{\mathbf{a} - c\mathbf{1}}{2}$$

Note, under optimal prices, all agents purchase a postiive amount

Immediate corollary: If G is symmetric, optimal prices are

$$\mathbf{p} = \frac{\mathbf{a} + c\mathbf{1}}{2},$$

independent of the network structure

Optimal Pricing

Recall the Katz-Bonacich centralities $\mathcal{K}(G,\alpha) = (I - \alpha G)^{-1}\mathbf{1}$

Theorem

Assume consumers are symmetric, $a_i = a$ and $b_i = b$ for all i. The optimal prices are

$$\mathbf{p} = \frac{a+c}{2}\mathbf{1} + \frac{a-c}{8b}\left[G\mathcal{K}\left(\frac{G+G^T}{2}, \frac{1}{2b}\right) - G^T \quad \frac{G+G^T}{2}, \frac{1}{2b}\right)\right]$$

Base price plus markup (influence by others) minus discount (influence to others)

Importance of Knowing the Network

Compare optimal prices ignoring the network to optimal prices with perfect information

- Π_0 profit assuming $g_{ij} \equiv 0$
- Π_N optimal profit with network information

Theorem

Assume players are symmetric, and define $M = \Lambda - G$ *and* $\tilde{M} = \frac{MM^{-T} + M^T M^{-1}}{4}$. *Then,*

$$\frac{1}{2} + \lambda_{min}\left(\tilde{M}\right) \leq \frac{\Pi_0}{\Pi_N} \leq \frac{1}{2} + \lambda_{max}\left(\tilde{M}\right)$$

From corollary, we know if $G = G^T$, then $\Pi_0 = \Pi_N$; value of network information increases with asymmetry of interactions