Recitation 1: Utility Maximization

Maddie McKelway & Will Rafey

Overview

1. Utility Maximization: The Basics
2. Utility Maximization over Two Goods
3. Utility Maximization over Two Periods
4. Utility Maximization over Three Periods

Utility and Diminishing Marginal Utility

- Utility: the satisfaction from consuming a good or service
  - Utility function $u : X \to \mathbb{R}$.
- Marginal utility: the additional satisfaction from consuming one more unit of the good or service
- Law of diminishing marginal utility
  - The more you consume, the less utility you get from the additional unit.

The Budget Constraint and Utility Maximization

- We live in a scarce world; we face constraints on what and how much goods and services we can have
- Economics assumes that people maximize their utility functions subject to their constraints
- Today we will review utility maximization in traditional economic theory
- Behavioral economics considers whether these models are realistic and, if not, how they can be extended to be more realistic

Start with a Model of Two Goods

Suppose you are trapped on an island. There are only two kinds of plants that can be planted on the island: oranges and potatoes. The island has a cultivated area of 4 acres. Each acre can produce 1 unit of oranges *or* 1 unit of potatoes. How should you allocate the land between oranges and potatoes?

- Need a measure to compare different combinations of oranges and potatoes - use a utility function!
- Assume your utility function is

$$u(o, p) = \ln(o) + 2\ln(p)$$

Model of Two Goods (Continued)

- The constraints can be constructed from the information provided:

$$o + p \leq 4$$

  as well as $o, p \in [0, 4]$
  - In this example, the prices of oranges and potatoes are the same – both require 1 acre of land for 1 unit of output
  - Can drop $o, p \in [0, 4]$
    - Lower bound implied by log utility and upper bound implied by $o + p \leq 4$
- The problem becomes

$$\max_{o,p} \ln(o) + 2\ln(p)$$
$$s.t. \quad o + p \leq 4$$

Solving the Math

- There are many ways to solve constrained maximization problems
- A common method used in economics is the Lagrangian method
- Another is to equate the ratios of marginal utilities to prices
  - If (x^*, y^*) is an interior solution to the maximization problem

$$\max_{x,y} u(x,y)$$
$$s.t. \qquad p_1 x + p_2 y \leq w$$

  then

$$\frac{MU_x}{p_1}\Big|_{(x^*,y^*)} = \frac{MU_y}{p_2}\Big|_{(x^*,y^*)}$$

  - Combining this with the requirement that the solution lie on the budget constraint gives (x^*, y^*)
    - "Don't leave money on the table"
    - i.e. $p_1 x^* + p_2 y^* = w$

Graphical Interpretation

Image removed due to copyright restrictions.
View Fig. 3.6 Utility Maximization.

Solving the Math in Our Example

- Equating the ratios of marginal utilities to prices gives

$$\frac{\partial u}{\partial o} = \frac{\partial u}{\partial p}$$

$$\frac{1}{o^*} = \frac{2}{p^*}$$

- Assuming the solution lies on the budget constraint gives

$$o^* + p^* = 4$$

- Combining gives the solution

$$o^* = \frac{4}{3},\ p^* = \frac{8}{3}$$

Two Periods

Now suppose there are two periods and no oranges.

You begin period 1 with 4 units of potatoes. You cannot grow any more and you have no other source of food for the two periods.

This means that in period 1 you have to save food for period 2. You can store the potatoes in a basket between the periods, but in period 2, only 80% of the saved potatoes will remain (the rest will be eaten by mice!).

Discounting Future Consumption

- Two goods become consumption in period 1 (c_1) and consumption in period 2 (c_2)
- Now the utility function will include temporal discounting
  - Why? People may not value current and future consumption the same
- Utility becomes

$$u(c_1, c_2) = \ln(c_1) + \delta \ln(c_2)$$

  - δ is called the "discount factor" and captures intertemporal preferences
  - Generally we assume $\delta \leq 1$
  - Larger $\delta \Rightarrow$ more patient

Solving the Math

- The constraints are

$$c_2 \leq 0.8(4 - c_1)$$

  as well as $c_1 \in [0, 4]$, $c_2 \in [0, 3.2]$
  - $c_1 \in [0, 4]$, $c_2 \in [0, 3.2]$ again implied
  - We rewrite constraint as $0.8c_1 + c_2 \leq 3.2$
    - As if we have prices $(p_1, p_2) = (0.8, 1)$

- Equating the ratios of marginal utilities to prices gives

$$\frac{\partial u/\partial c_1}{p_1} = \frac{\partial u/\partial c_2}{p_2}$$

$$\frac{1}{c_1^*} = \frac{0.8\delta}{c_2^*}$$

Solving the Math (Continued)

- Combining with $c_2^* = 0.8(4 - c_1^*)$ gives

$$c_1^* = \frac{4}{1+\delta}, \; c_2^* = \frac{3.2\delta}{1+\delta}$$

- Comparative statics
  - More patient (i.e., higher δ) $\implies$ more c_2, less c_1

Three Periods

Bad news! Your Amazon Prime membership has lapsed and now you must rely on potatoes for three periods.

From period 2 to period 3, the mice will again eat 20% of the remaining potato stock.

Now at period 1, you also value period 3 consumption (c_3) but value it even less than you do period 2 consumption.

How should you allocate your consumption across periods?

Exponential Discounting

- Paul Samuelson (MIT) proposed using the same discount factor on future utility from each period to the next

$$U(c_1, c_2, ..., c_T) = u(c_1) + \delta u(c_2) + \delta^2 u(c_3) + ... + \delta^{T-1} u(c_T)$$

  - Here $u(c_t)$ is the per-period utility, and $U(\cdot)$ specifies how people value consumption into the future at $t = 1$

- In a three-period model, we consider

$$U(c_1, c_2, c_3) = \ln(c_1) + \delta \ln(c_2) + \delta^2 \ln(c_3)$$

- Is this a realistic assumption?

Solving the Math

- The problem becomes

$$\max_{c_1, c_2, c_3 \geq 0} \ln(c_1) + \delta \ln(c_2) + \delta^2 \ln(c_3)$$
$$s.t. \quad (i) \quad c_2 \leq 0.8(4 - c_1)$$
$$(ii) \quad c_3 \leq 0.8[0.8(4 - c_1) - c_2]$$

  as well as $c_t \in [0, 0.8^{t-1} \times 4]$

- $c_t \in [0, 0.8^{t-1} \times 4]$ is implied. Since $c_3 \geq 0$, (ii) implies (i). So we can use just (ii) and can rewrite it as

$$0.64c_1 + 0.8c_2 + c_3 \leq 2.56$$

Solving the Math

- The interior solution (c_1^*, c_2^*, c_3^*) satisfies

$$\frac{\frac{\partial U(c_1,c_2,c_3)}{\partial c_1}}{0.64} = \frac{\frac{\partial U(c_1,c_2,c_3)}{\partial c_2}}{0.8} = \frac{\frac{\partial U(c_1,c_2,c_3)}{\partial c_3}}{1}$$

$$\frac{1}{c_1^*} = \frac{0.8\delta}{c_2^*} = \frac{0.64\delta^2}{c_3^*}$$

- Combining with $0.64c_1^* + 0.8c_2^* + c_3^* = 2.56$, we get

$$c_1^* = \frac{4}{1+\delta+\delta^2}, \ c_2^* = \frac{3.2\delta}{1+\delta+\delta^2}, \ c_3^* = \frac{2.56\delta^2}{1+\delta+\delta^2}$$

- Note that the ratio of consumption across periods is the same! (As long as per-period utility and price ratios are the same.) This is the essence of exponential discounting.

Time Consistency

- Suppose you follow the optimal allocation plan and consume $c_1^* = \frac{4}{1+\delta+\delta^2}$ in period 1. Then in period 2, will you deviate from consuming $c_2^* = \frac{3.2\delta}{1+\delta+\delta^2}$?

- In period 2, there remains $0.8\frac{4(\delta+\delta^2)}{1+\delta+\delta^2}$ potatoes. Now you are facing a 2-period problem. As shown above, in a 2-period problem, you will consume fraction $\frac{1}{1+\delta}$ of the total in the first period and leave the rest for the second period.

$$\frac{1}{1+\delta} \cdot \frac{0.8 \cdot 4(\delta + \delta^2)}{1+\delta+\delta^2} = \frac{3.2\delta}{1+\delta+\delta^2}$$

- This is exactly c_2^*

- Is this a coincidence? No! Exponential discounting assumes the same discount factor between every future period to the next.

Further References

- *Microeconomics* by Pindyck and Rubinfeld
- 14.03 MIT OpenCourseware: see notes on class website