MIT 14.13 – Problem Set 2

Please make sure to explain your answers carefully and concisely, i.e. do not simply write a numeric answer without an explanation of how you arrived at this answer. Answers without adequate explanation will not receive full credit.

Question: Payday Lending

Calvin is a fully naive hyperbolic discounter with $\beta = 0.5$ and $\delta = 1$ and $\hat{\beta} = 1$. Hobbes is a fully sophisticated hyperbolic discounter with $\beta = 0.5$ and $\delta = 1$ and $\hat{\beta} = \beta = 0.5$. They lives for three periods: $t = 0$, 1, and 2. They derive utility from consumption in each period. They have the same instantaneous CRRA utility from consuming an amount $c_t \geq 0$ (i.e. $c_t < 0$ is not possible) in period t :

$$u(c_t) = \sqrt{c_t} \text{ for } t = 0, 1, 2$$

Accordingly, their discounted lifetime utility from the perspective of period $t = 0, 1$ is given by

$$U_t(c_0, c_1, c_2) = \sqrt{c_t} + \beta \sum_{s=t+1}^{s=2} \sqrt{c_s}$$

and their discounted lifetime utility at $t = 2$ is simply $\sqrt{c_2}$.

We also define their long-term lifetime utility as

$$W(c_0, c_1, c_2) = \sqrt{c_0} + \sqrt{c_1} + \sqrt{c_2}$$

which for instance captures their discounted lifetime utility from a period preceding 0, without distortion from present bias.

Calvin and Hobbes start with wealth of $e_0 = \$1500$ at $t = 0$. They can keep their wealth in a checking account, which has no interest and would allow them to withdraw money at any time. That is, if they put $\$x$ into their account in period 0, they could withdraw up to $\$x$ at period 1. Similarly, if they put $\$x$ into their account in period 1, they could withdraw up to $\$x$ at period 2.

They receive a paycheck of $y = \$1200$ at $t = 2$, which is known and perfectly anticipated by both of them at all times.

Finally, they have access to a payday lending service: they can borrow up to \$600 in period 1, but they have to repay twice the borrowed amount on their payday in period 2 (i.e., they can borrow with an interest rate of 100% between these two periods).

1. Let's first consider a third fictional character, Susie, who is not present biased, and does not discount the future. Her utility at any period $t = 0, 1, 2$ is $\sum_{s=t}^{s=2} \sqrt{c_s}$ Susie also starts with \$1500 at $t = 0$, has access to the checking account, and anticipates receiving \$1200 in period 2, but has no access to payday lending. Derive Susie's consumption in period 0, 1 and 2. In particular, show that Susie does not use the checking account from period 1 to period 2.

Solution: Susie's consumption path is $c_0 = c_1 = 750, c_2 = 1200$.
Susie is fully patient: she puts equal weight on each period. Her instantaneous utility is strictly concave. Thus, she wants to smooth consumption as much as possible. Ideally, she would consume a third of her lifetime income at each period. But she cannot consume less than \$1200 in period 2 since she has no access to credit. \$1200 is more than a third of her lifetime income (\$2700) therefore she will consume exactly \$1200 in period 2 and not use the checking account between period 1 and 2. She will spread the remaining lifetime wealth (\$1500) equally between period 0 and 1 using the checking account.

2. Explain (with no formal derivation) why this means that neither Calvin nor Hobbes would use the checking account from period 1 to period 2.

Solution: Calvin and Hobbes have preferences which add present bias on top of Susie's preferences. Given that Susie does not use her checking account from period 1 to period 2, Calvin and Hobbes would not either.

This intuition is derived formally below, where we find non-negative amounts borrowed from the payday lending.

Given this result, we will now work under the (non-binding) assumption that the checking account is only available from period 0 to period 1, for simplicity.

3. Let $e_1 \geq 0$ denote the amount of money in Calvin's (or Hobbes') checking account when he enters period 1. Assume $e_1 \leq y$. Derive the amount that he decides to borrow from the payday lending service, b, as a function of e_1. Show that he will consume an equal amount in periods 1 and 2, i.e. $c_1 = c_2$.

Solution:
In period 1, our character decides how much to borrow in order to maximize $\sqrt{c_1} + \beta\sqrt{c_2}$. Consumption in period 1 is equal to the amount in our character's bank account plus the amount borrowed, so $c_1 = e_1 + b$. Consumption in period 2 is equal to income minus twice the borrowed amount (since the payday lending interest rate is 100%), so $c_2 = y - 2b$. Our character thus solves:

$$\max \sqrt{c_1} + \beta\sqrt{c_2} = \max_{b \geq 0} \sqrt{e_1 + b} + \beta\sqrt{y - 2b}$$

Differentiating with respect to b, we get the first-order condition:

$$\frac{1}{2}\frac{1}{\sqrt{e_1 + b}} - \frac{\beta}{2}\frac{2}{\sqrt{y - 2b}} = 0$$

Plugging in $\beta = \frac{1}{2}$ and solving for b,

$$\frac{1}{2}\frac{1}{\sqrt{e_1 + b}} = \frac{1}{2}\frac{\frac{1}{2}(2)}{\sqrt{y - 2b}}$$

$$\frac{1}{\sqrt{e_1 + b}} = \frac{1}{\sqrt{y - 2b}}$$

$$\sqrt{e_1 + b} = \sqrt{y - 2b}$$

$$e_1 + b = y - 2b$$

Since $c_1 = e_1 + b$ and $c_2 = y - 2b$, we've just shown that $c_1 = c_2$ optimally, as desired. Continuing,

$$y - 3b = e_1$$

$$3b = y - e_1$$

$$b = \frac{y - e_1}{3}$$

Which will only be valid for $y \geq e_1$. For higher e_1, the character would not borrow anything.
As a result, in period 1, Calvin (and Hobbes) will consume $c_1 = e_1 + b = \frac{y+2e_1}{3}$. In period 2, they will consume $c_2 = y - 2b = \frac{y+2e_1}{3} = c_1$. This is coming from the fact that utility is discounted at rate $\beta = \frac{1}{2}$ in period 2 relative to period 1, but money can be transferred from period 2 to period 1 at the conversion rate $\frac{1}{1+r} = \frac{1}{2}$.

4. Using the result from the previous question, derive the amount of money e_1^S that Hobbes, who is fully sophisticated, decides to put in his checking account in period 0. *Hint: Do not worry about checking corner solutions, i.e. assume that* $e_1^S \leq y$ *in order to use the answer to the previous question, and just verify that the value obtained indeed verifies this inequality.*

Solution:

Hobbes is fully sophisticated, so in period 0, he correctly anticipates how b will depend on e_1. Consumption in period 0 is his initial wealth minus the amount he puts in the bank account, so $c_0 = e_0 - e_1$. From question 3, we know that $c_1 = c_2 = \frac{y+2e_1}{3}$. Since e_0 and y are fixed, his choice of e_1 completely determines his consumption path $\{c_0, c_1, c_2\}$. So he solves:

$$\max \sqrt{c_0} + \beta\sqrt{c_1} + \beta\sqrt{c2} = \max_{e_1 \geq 0} \sqrt{e_0 - e_1} + 2\beta\sqrt{\frac{y + 2e_1}{3}}$$

Differentiating with respect to e_1, we get the first-order condition:

$$-\frac{1}{2\sqrt{e_0 - e_1}} + 2\beta\frac{\frac{2}{3}}{2\sqrt{\frac{y+2e_1}{3}}} = 0$$

Plugging in $\beta = \frac{1}{2}$ and solving for e_1,

$$\frac{1}{2\sqrt{e_0 - e_1}} = \frac{2(\frac{1}{2})(\frac{2}{3})}{2\sqrt{\frac{y+2e_1}{3}}}$$

$$\frac{1}{2\sqrt{e_0 - e_1}} = \frac{1}{3\sqrt{\frac{y+2e_1}{3}}}$$

$$2\sqrt{e_0 - e_1} = 3\sqrt{\frac{y + 2e_1}{3}}$$

$$4(e_0 - e_1) = 9\left(\frac{y}{3} + \frac{2e_1}{3}\right)$$

Solving the last line for e_1 gives:

$$e_1^S = \frac{4e_0 - 3y}{10} = \frac{4(1500) - 3(1200)}{10} = 240$$

We verify that e_1^S thus obtained is lower than y and yields a positive b.

5. How much will Hobbes end up borrowing from the payday lending service and consuming in each period?

Solution:

Using the results from questions 3 and 4, we obtain:

$$c_0 = e_0 - e_1 = 1260$$

$$b = \frac{y - e_1}{3} = \frac{1200 - 240}{3} = 320$$

$$c_1 = c_2 = \frac{y + 2e_1}{3} = \frac{1200 + 2(240)}{3} = 560.$$

6. Now, let's consider Calvin, who is fully naive ($\hat{\beta} = 1$). In period 0, how much does Calvin *predict* he will borrow from the payday lending service in period 1 if he were to leave e_1 in his checking account?

Solution:

Calvin is fully naive, so in period 0, he predicts that he will behave in period 1 as if $\beta = b\hat{e}ta = 1$. Let's call Calvin's predicted borrowing $\hat{b}$. We still have that $c_1 = e_1 + \hat{b}$ and $c_2 = y - 2\hat{b}$. So he predicts that in period 1, he will solve the following problem

$$\max \sqrt{c_1} + \beta\sqrt{c_2} = \max \sqrt{c_1} + \sqrt{c_2} = \max_{\hat{b} \geq 0} \sqrt{e_1 + \hat{b}} + \sqrt{y - 2\hat{b}}$$

Differentiating with respect to b, we get the first-order condition:

$$\frac{1}{2}\frac{1}{\sqrt{e_1 + \hat{b}}} - \frac{1}{2}\frac{2}{\sqrt{y - 2\hat{b}}} = 0$$

Solving for b,

$$\frac{1}{2}\frac{1}{\sqrt{e_1 + \hat{b}}} = \frac{1}{2}\frac{2}{\sqrt{y - 2\hat{b}}}$$

$$\frac{1}{2}\frac{1}{\sqrt{e_1 + \hat{b}}} = \frac{1}{\sqrt{y - 2\hat{b}}}$$

$$\sqrt{y - 2\hat{b}} = 2\sqrt{e_1 + \hat{b}}$$

$$y - 2\hat{b} = 4(e_1 + \hat{b})$$

Rearranging, we get:

$$\hat{b} = \frac{y - 4e_1}{6}$$

Thus Calvin predicts that he will borrow $\hat{b} = \frac{y - 4e_1}{6}$ if $y \geq 4e_1$ and 0 otherwise.

7. Derive the amount e_1^N that Calvin decides to leave in his checking account in period 0. *Hint: Do not worry about checking corner solutions, i.e. assume that $4e_1^N \leq y$ in order to use the answer to the previous question, and just verify that the value obtained indeed verifies this inequality.*

Solution:

Calvin is fully naive, so he predicts that his borrowing behavior will be as found above in question 6. However, like Hobbes, in period 0, he discounts both period-1 utility and period-2 utility by β. He thus solves:

$$\max \sqrt{c_0} + \beta\sqrt{c_1} + \beta\sqrt{c_2}$$

$$\max_{e_1 \geq 0} \sqrt{e_0 - e_1} + \beta\sqrt{e_1 + \hat{b}} + \beta\sqrt{y - 2\hat{b}}$$

$$\max_{e_1 \geq 0} \sqrt{e_0 - e_1} + \beta\sqrt{e_1 + \frac{y - 4e_1}{6}} + \beta\sqrt{y - 2\left(\frac{y - 4e_1}{6}\right)}$$

$$\max_{e_1 \geq 0} \sqrt{e_0 - e_1} + \beta\sqrt{\frac{y}{6} + \frac{e_1}{3}} + \beta\sqrt{\frac{2y}{3} + \frac{4e_1}{3}}$$

$$\max_{e_1 \geq 0} \sqrt{e_0 - e_1} + \beta\sqrt{\frac{y}{6} + \frac{e_1}{3}} + \beta\sqrt{4\left(\frac{y}{6} + \frac{e_1}{3}\right)}$$

$$\max_{e_1 \geq 0} \sqrt{e_0 - e_1} + \beta\sqrt{\frac{y}{6} + \frac{e_1}{3}} + 2\beta\sqrt{\frac{y}{6} + \frac{e_1}{3}}$$

$$\max_{e_1 \geq 0} \sqrt{e_0 - e_1} + 3\beta\sqrt{\frac{y + 2e_1}{6}}$$

Differentiating with respect to e_1, we get the first-order condition:

$$\frac{1}{2}\frac{1}{\sqrt{e_0 - e_1}} - 3\beta\frac{1}{2}\frac{\frac{2}{6}}{\sqrt{\frac{y+2e_1}{6}}}$$

Plugging in $\beta = \frac{1}{2}$ and solving for e_1,

$$\frac{1}{2}\frac{1}{\sqrt{e_0 - e_1}} = \frac{1}{2}\frac{3(\frac{1}{2})(\frac{1}{3})}{\sqrt{\frac{y+2e_1}{6}}}$$

$$\frac{1}{\sqrt{e_0 - e_1}} = \frac{1}{2}\frac{1}{\sqrt{\frac{y+2e_1}{6}}}$$

$$2\sqrt{\frac{y + 2e_1}{6}} = \sqrt{e_0 - e_1}$$

$$4\left(\frac{y + 2e_1}{6}\right) = e_0 - e_1$$

Solving the last line for e_1 gives

$$e_1^N = \frac{6e_0 - 4y}{14} = \frac{6(1500) - 4(1200)}{14} = 300$$

We verify that e_1^N thus obtained is non greater than $y/4$ and yields a non-negative $\hat{b}$.

8. How much will Calvin end up borrowing from the payday lending service and consuming in each period?

Solution:

In period 0, Calvin consumes $e_0 - e_1 = 1200$. Then, Calvin surprises himself in period 1 and borrows according to the behavior with $\beta = \frac{1}{2}$ that we derived in question 3: $b = \frac{y-e_1}{3} = 300$. So while he expects not to borrow,

$$\hat{b}(300) = \frac{y - 4(300)}{6} = \frac{1200 - 1200}{6} = 0,$$

he ends up doing so. He then consumes

$$c_1 = c_2 = \frac{y + 2e_1}{3} = \frac{1200 + 2(300)}{3} = 600.$$

9. Discuss how Calvin's and Hobbes' consumption paths differ. Compute their long-term lifetime utilities, compare them, and discuss intuitively why they are ordered in this way.

Solution:

Calvin's consumption path is: $c_0 = 1200$, $c_1 = 600$, and $c_2 = 600$. Hobbes' consumption path is: $c_0 = 1260$, $c_1 = 560$, and $c_2 = 560$.

Calvin is naive about his future present bias and thus underestimates his propensity to borrow from the payday lender. Thinking that he won't borrow in period 1, Calvin thus saves more and consumes less in period 0. He then surprises himself in period 1 by borrowing from the payday lender.

Hobbes is sophisticated and realizes that he will borrow in period 1. Since this borrowing will increase his period 1 consumption, Hobbes saves less and consumes more in period 0. Hobbes thus borrows less than Calvin in period 1.

Calvin's long-term lifetime utility is $\sqrt{1200} + \sqrt{600} + \sqrt{600} \approx 83.63$. Hobbes' long-term lifetime utility is $\sqrt{1260} + \sqrt{560} + \sqrt{560} \approx 82.83$.

Despite being naive, Calvin attains higher lifetime utility! Being naive has two consequences here: 1) Calvin borrows more in period 1 and incurs more interest costs, and 2) Calvin's consumption path is actually smoother, since he saves more in period 0. The second effect outweighs the first. Hobbes' sophistication lowers his benefit of saving in period 0 (because he anticipates that borrowing in period 1 is already pushing up his period-1 consumption). So Hobbes saves less in period 0, and his resulting consumption path is less smooth than Calvin's.

10. Now, assume that no payday lending service is available. Derive the amounts left in the checking account in period 0 by Calvin and Hobbes.

Solution:

Without payday lending service, there is only one decision, made in period 0: how much to leave in the checking account. Sophistication thus does not matter here (c_2 is fixed at y) and both characters solve the same problem:

$$\max_{e_1 \geq 0} \sqrt{e_0 - e_1} + \beta\sqrt{e_1} + \beta\sqrt{y}$$

which yields the first-order condition:

$$-\frac{1}{2\sqrt{e_0 - e_1}} + \frac{\beta}{2\sqrt{e_1}} = 0$$

Plugging in $\beta = \frac{1}{2}$ and solving for e_1,

$$\frac{1}{2}\frac{1}{\sqrt{e_0-e_1}} = \frac{1}{2}\frac{\frac{1}{2}}{\sqrt{e_1}}$$

$$2\sqrt{e_1} = \sqrt{e_0-e_1}$$

$$4e_1 = e_0 - e_1$$

$$5e_1 = e_0$$

$$e_1 = \frac{e_0}{5} = \frac{1500}{5} = 300$$

11. Derive the full consumption paths of Calvin and Hobbes in the absence of payday lending. Compare their long-term lifetime utilities to the values found in question 9. Discuss this comparison.

Solution:

From the previous question, both characters have the same consumption path: $c_0 = 1200$, $c_1 = 300$ and $c_2 = 1200$. This yields long-term lifetime utility of $\sqrt{300} + 2\sqrt{1200} \approx 86.60$.

In question 9 with payday lending, we found that Calvin's lifetime utility was about 83.63 and Hobbes' lifetime utility was about 82.83. So both characters are helped by the elimination of payday lending. Calvin benefits because he can no longer borrow and thus no longer incurs interest costs. Hobbes is helped because he no longer anticipates borrowing in period 1. This raises the the benefit for Hobbes of saving in period 0 (he no longer anticipates that c_1 will be pushed up by borrowing), so he saves more in period 0 and attains a smoother consumption path, raising his lifetime utility.

12. Suppose that in period 0, a referendum is organized to ask Calvin and Hobbes whether they want the government to implement a policy that shuts down payday lending. The policy would require some administrative costs which would result in a tax of \$1 levied at the end of period 1. The two options to vote for are Yes and No. What would Calvin vote? What would Hobbes vote? (assuming they are both selfish and only care about improving their own utility). Discuss what this example suggests for the real world problem of regulating payday lending.

Solution:

Calvin would be helped by the removal of payday lending, but he fails to realize this since he plans not to borrow anything. He would thus vote No since the policy would bring him nothing and cost him a dollar of tax. Hobbes would also benefit from payday lending, and since he is sophisticated he actually foresees this benefit in period 0. The increase in Hobbes' lifetime utility resulting from the elimination of payday lending is much larger than the utility cost of paying \$1 in period 1. Hobbes would thus vote Yes. This example illustrates that regulating payday lending may be difficult, since not all people who would benefit from such regulation are actually aware of the benefits.

For the rest of the problem, we consider a world where a shock just hurt Calvin and Hobbes before the time analyzed in the problem, so that their initial wealth is now $e_0 = \$200$. Note that the conclusions from question 2 also apply here so it's still correct to simply assume that there is no checking account from period 1 to period 2.

13. Noting that answers to questions 3 and 6 are unchanged, show that neither Calvin nor Hobbes leave anything in their checking accounts in period 0.

Solution:

The objectives in questions 4 and 7 remain the same. Using the formulas we found expressing e_1 as a function of e_0 and y, we obtain:

$$e_1^S = \frac{4e_0 - 3y}{10} = -280$$
$$e_1^N \frac{6e_0 - 4y}{14} \approx -257$$

Since saving must be non-negative, the period-0 optimization problems are now solved at the corner solution $e_1 = 0$ for both Calvin and Hobbes.

14. Compute Calvin and Hobbes' resulting consumption path.

Solution:

With $e_1 = 0$ for both characters, and with the logic from question 3 still mandating that $c_1 = c_2$, we obtain the consumption path

$$c_0 = e_0 = 200$$
$$c_1 = c_2 = \frac{y + 2e_1}{3} = \frac{y + 0}{3} = \frac{1200}{3} = 400$$

for both characters.

15. As in question 10, derive the amounts left in the checking accounts in period 0 by Calvin and Hobbes if no payday lending service is available. Derive the resulting consumption paths.

Solution:

If there is no payday lending, as in question 10, $e_1 = \frac{e_0}{5} = 40$. This results in the consumption path $c_0 = e_0 - 40 = 160$, $c_1 = 40$, $c_2 = y = 1200$.

16. Compare the long-term lifetime utilities of Calvin and Hobbes with and without access to the payday lending service now that their initial wealth e_0 is lower. Why is this comparison yielding a different conclusion than in question 11?

Solution:

With access to the payday lending, both characters attain long-term lifetime utility of $\sqrt{200}+2\sqrt{400} \approx 54.14$. Without payday lending, they both attain long-term lifetime utility of $\sqrt{160} + \sqrt{40} + \sqrt{1200} \approx 53.61$. Thus both characters are now harmed by the elimination of payday lending, whereas both were helped in question 11.

With a negative shock to initial wealth, there is a very large return from the ability to smooth consumption by borrowing against period 2, even at the very high payday lending rate. The inability to borrow against period 2 when e_0 is low thus causes the elimination of payday lending to harm both characters. Since Hobbes wasn't planning to borrow in period 1 anyway, the elimination of payday lending doesn't provide any consumption-smoothing benefits (whereas it did provide such benefits in question 11).