14.123 Problem Set 1 Solution
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Q1. Let P be the set of all lotteries $p = (p_x, p_y, p_z)$ on a set $C = \{x, y, z\}$ of consequences. Below, you are given pairs of indifference sets on P. For each pair, check whether the indifference sets belong to a preference relation that has a Von-Neumann and Morgenstern representation (i.e. expected utility representation). If the answer is Yes, provide a Von-Neumann and Morgenstern utility function; otherwise show which Von-Neumann and Morgenstern axiom is violated. (In the figures below, setting $p_z = 1 - p_x - p_y$, we describe P as a subset of $\mathbb{R}^2$.)

(a) $I_1 = \{p|1/2 \leq p_y \leq 3/4\}$ and $I_2 = \{p|p_y = 1/4\}$:

No, the Independence Axiom is violated. I'll use (2.2) from Question 2. Take $(1/4, 3/4), (1/2, 1/2) \in I_1$ and $a = 2$. From $(1/4, 3/4) \sim (1/2, 1/2)$, we have

$$\begin{aligned}(1/4, 3/4) =& 2(1/4, 3/4) + (-1)(1/4, 3/4) \\ \sim& 2(1/2, 1/2) + (-1)(1/4, 3/4) = (3/4, 1/4),\end{aligned}$$

which is a contradiction to $(3/4, 1/4) \in I_2$.

(b) $I_1 = \{p|p_y = p_x\}$ and $I_2 = \{p|p_y = p_x + 1/2\}$:

Yes, an example is $U(p) = p_x - p_y$.

Q2. For any preference relation that satisfies the Independence Axiom, show that the following are true.

(a) For any $p, q, r, r' \in P$ with $r \sim r'$ and any $a \in (0, 1]$,

$$ap + (1 - a)r \succeq aq + (1 - a)r' \Leftrightarrow p \succeq q. \tag{1}$$

$r \sim r'$ implies that $r \succeq r'$ and $r' \succeq r$. From the Independence Axiom, for any $a \in (0, 1]$,

$$p \succeq q \Longleftrightarrow ap + (1 - a)r \succeq aq + (1 - a)r.$$

The Independence Axiom also implies that

$$\begin{aligned}aq + (1 - a)r &\succeq aq + (1 - a)r', \\ aq + (1 - a)r' &\succeq aq + (1 - a)r,\end{aligned}$$

and we have

$$ap+(1-a)r \succeq aq+(1-a)r \Longrightarrow ap+(1-a)r \succeq aq+(1-a)r',$$
$$ap+(1-a)r \succeq aq+(1-a)r' \Longrightarrow ap+(1-a)r \succeq aq+(1-a)r$$

by transitivity.

$$p \succeq q \Longleftrightarrow ap+(1-a)r \succeq aq+(1-a)r'.$$

(b) For any $p, q, r \in P$ and any real number a such that $ap+(1-a)r, aq+(1-a)r \in P$,

$$\text{if } p \sim q, \text{ then } ap+(1-a)r \sim aq+(1-a)r. \tag{2}$$

The case $a \in (0,1]$ is given by the Independence Axiom, and the case $a=0$ always holds from $r \sim r$.

For $a > 1$, $1/a \in (0,1]$, and the Independence Axiom gives that

$$\begin{aligned} & ap+(1-a)r \sim aq+(1-a)r \\ \Longleftrightarrow & \frac{1}{a}(ap+(1-a)r)+\frac{a-1}{a}r \sim \frac{1}{a}(aq+(1-a)r)+\frac{a-1}{a}r \\ \Longleftrightarrow & p \sim q. \end{aligned}$$

For $a < 0$, $1/(1-a) \in (0,1]$, and if $p \sim q$,

$$\begin{aligned} \frac{1}{1-a}(ap+(1-a)r)+\frac{-a}{1-a}q &\sim \frac{1}{1-a}(ap+(1-a)r)+\frac{-a}{1-a}p \\ &\sim r \\ &\sim \frac{1}{1-a}(aq+(1-a)r)+\frac{-a}{1-a}q. \end{aligned}$$

By the Independence Axiom, we have

$$ap+(1-a)r \sim aq+(1-a)r.$$

Therefore, for any $a \in \mathbb{R}$ such that $ap+(1-a)r, aq+(1-a)r \in P$,

$$\text{if } p \sim q, \text{ then } ap+(1-a)r \sim aq+(1-a)r.$$

(c) For any $p, q \in P$ with $p \succ q$ and any $a, b \in [0,1]$ with $a > b$,

$$ap+(1-a)q \succ bp+(1-b)q. \tag{3}$$

If $b = 0$, the Independence Axiom gives that

$$ap + (1-a)q \succ aq + (1-a)q \sim q.$$

For $b > 0$, we have $b/a \in (0,1)$, and

$$\begin{aligned} & ap + (1-a)q \succ q \\ \Longrightarrow\ & ap + (1-a)q \sim \frac{b}{a}(ap + (1-a)q) + \frac{a-b}{a}(ap + (1-a)q) \\ & \qquad \succ \frac{b}{a}(ap + (1-a)q) + \frac{a-b}{a}q \sim bp + (1-b)q. \end{aligned}$$

(d) There exist $c^B, c^W \in C$ such that for any $p \in P$,

$$c^B \succeq p \succeq c^W. \tag{4}$$

[Hint: use completeness and transitivity to find $c^B, c^W \in C$ with $c^B \succeq c \succeq c^W$ for all $c \in C$; then use induction on the number of consequences and the Independence Axiom.]

The set of consequences C is finite. Let n be the number of consequences. When $n = 1$, $c^B \sim p \sim c^W$ for all $p \in P$.

Suppose that for $n = k$, there exist $c^B, c^W \in C$ such that for any $p \in P$,

$$c^B \succeq p \succeq c^W. \tag{$*$}$$

Consider $n = k+1$. Let $C = \{c_1, \cdots, c_{k+1}\}$ and $C' = \{c_1, \cdots, c_k\}$. From ($*$), there exist $c^{B'}, c^{W'} \in C'$ such that $c^{B'} \succeq p' \succeq c^{W'}$. If $c_{k+1} \succ c^{B'}$, let $c^B = c_{k+1}$, $c^W = c^{W'}$. If $c^{W'} \succ c_{k+1}$, let $c^B = c^{B'}$, $c^W = c_{k+1}$. Otherwise, $c^B = c^{B'}$, $c^W = c^{W'}$. Any $p \in P$ can be written as $p = ap' + (1-a)c_{k+1}$ for some $a \in [0,1]$ and a lottery p' over $C' = \{c_1, \cdots, c_k\}$.

We have $c^B \succeq p', c_{k+1} \succeq c^W$, and by the Independence Axiom,

$$\begin{aligned} c^B & \succeq ap' + (1-a)c^B \\ & \succeq ap' + (1-a)c_{k+1} = p \\ & \succeq ap' + (1-a)c^W \\ & \succeq c^W. \end{aligned}$$

Q3. Let P be the set of probability distribution on $C = \{x, y, z\}$. Find a continuous preference relation $\succeq$ on P, such that the indifference sets are all straight lines, but $\succeq$ does not have a von Neumann-Morgenstern utility representation.

Consider a preference relation represented by the following utility function

$$U(p_x, p_y, p_z) = \frac{p_y}{2 - p_x}.$$

$\succeq$ is complete, transitive and continuous, and the indifference set are straight lines, but the Independence Axiom is not satisfied.

Q4. Let $\succeq$ be the "at least as likely as" relation defined between events in Lecture 3. Show that $\succeq$ is a qualitative probability.

From $P1$, $\succeq$ is a preference relation, which implies that it's complete and transitive.

The second part follows from

$$\begin{aligned} B \succeq C &\iff f_B^{x,x'} \succeq f_C^{x,x'} \text{ for some } x, x' \in C, x \succ x' \\ &\iff f_{B\cup D}^{x,x'} \succeq f_{C\cup D}^{x,x'} && (\because P2) \\ &\iff B \cup D \succeq C \cup D. \end{aligned}$$

Lastly, from $P4$, there exists $x, x' \in C$ with $x \succ x'$. For any event B,

$$\begin{aligned} x \succ x' &\implies f_B^{x,x'} \succeq f_\emptyset^{x,x'} && (\because P2) \\ &\iff B \succeq \emptyset. \end{aligned}$$

Given any $x, x' \in C$ with $x \succ x'$, we have $f_S^{x,x'} \succeq f_\emptyset^{x,x'}$ from $P2$. There exist no $x, x' \in C$ with $x \succ x'$ such that $f_\emptyset^{x,x'} \succeq f_S^{x,x'}$.

$$S \succeq \emptyset, \emptyset \not\succeq S \implies S \succ \emptyset.$$