

11. Boundary conditions

This last example is particularly simple and in fact represents the same solution that would have been obtained using the linearized form of the exact two-dimensional equations (9.12) and (9.13). By assuming infinitely periodic solutions, we did not have to worry about applying boundary conditions when we inverted (10.5). But in general, it will be necessary to apply boundary conditions. A particular problem arises in applying conditions at horizontal boundaries at the top and/or bottom of fluid systems. Here, the standard Neumann or Dirichlet conditions would amount to a specification of either the geopotential, φ , or $\partial\varphi/\partial p$, which is proportional to temperature by the hydrostatic equation. Specifying φ at the surface would eliminate one of the important signals we are actually interested in predicting: atmospheric surface pressure and the geopotential height of the sea surface. Moreover, specifying temperature also negates the prediction of an important quantity. Therefore, typically we choose to actually *solve a predictive equation for temperature at a horizontal boundary*. Assuming that $\omega = 0$ on such a boundary, the appropriate *dynamical boundary condition* is (9.4):

$$D_g \frac{\partial\varphi}{\partial p} = \frac{-\bar{\alpha}\dot{Q}}{\theta}. \quad (11.1)$$

We shall see that much of the dynamics of atmospheric and oceanic quasi-balanced circulations enters through the dynamical boundary conditions (11.1).

But how should we think about the boundary conditions when inverting (10.5)? The Green's functions approach works well with strict Neumann or Dirichlet con-

ditions because image points can be used. But dynamical boundary conditions like (11.1) usually give space- and time-varying boundary conditions. But a trick borrowed from electrostatics is again useful: We replace the actual boundary temperature perturbation with a zero value, but add point “charges” of potential vorticity just inside the boundary, in analogy to the concept of bound charge. To see how this works, let us integrate (10.5) from a lower boundary at pressure p_0 to a short distance above the boundary:

$$\int_{p_0-\epsilon}^{p_0} q'_p dp = \int_{p_0-\epsilon}^{p_0} \frac{1}{f_0} \nabla^2 \varphi dp + \left. \frac{f_0}{\mathcal{S}} \frac{\partial \varphi}{\partial p} \right|_{p_0-\epsilon}^{p_0}. \quad (11.2)$$

Now in the limit of $\epsilon \rightarrow 0$, all the terms in (11.2) vanish because they are nonsingular. But suppose we *artificially* replace the actual temperature perturbation, $\partial\varphi/\partial p$, by zero at the boundary. Then (11.2) becomes

$$\int_{p_0-\epsilon}^{p_0} q'_p dp = - \left. \frac{f_0}{\mathcal{S}} \frac{\partial \varphi}{\partial p} \right|_{p_0-\epsilon}, \quad (11.3)$$

so that (11.2) can only be satisfied in this case if q'_p behaves like a delta function near the boundary.

From this development it follows that inverting (10.5) with an inhomogeneous boundary condition on temperature is equivalent to inverting it with a homogeneous boundary condition on temperature but inserting a delta function q'_p anomaly next to the boundary. From (11.3) and the hydrostatic equation (8.33), this delta func-

tion is

$$q'_p = \delta(p_0 - p) \times \begin{cases} \left. \frac{f_0}{S_0} \frac{R}{p_0} \theta' \right|_{p_0} & \text{atmosphere} \\ \left. \frac{f_0}{S_0} G_0 \sigma' \right|_{p_0} & \text{ocean,} \end{cases} \quad (11.4)$$

where δ is the delta function. This equivalence is valid at a *lower* boundary. At an *upper* boundary, a similar development leads to

$$q'_p = \delta(p - p_t) \times \begin{cases} \left. -\frac{f_0}{S_t} \frac{R}{p_t} \left(\frac{p_t}{p_0}\right)^{R/c_p} \theta' \right|_{p_t} & \text{atmosphere} \\ \left. -\frac{f_0}{S_t} G_t \sigma' \right|_{p_t} & \text{ocean,} \end{cases} \quad (11.5)$$

where p_t is the pressure on the upper boundary. *Note the minus signs in (11.5).*

This device is of enormous *conceptual* significance, because it tells us that boundary temperature anomalies behave like delta function q_p anomalies just inside the boundary. In the case of a lower boundary, the temperature anomalies behave like delta function q_p anomalies of the *same* sign, whereas at upper boundaries they behave like q_p anomalies of the *opposite* sign.

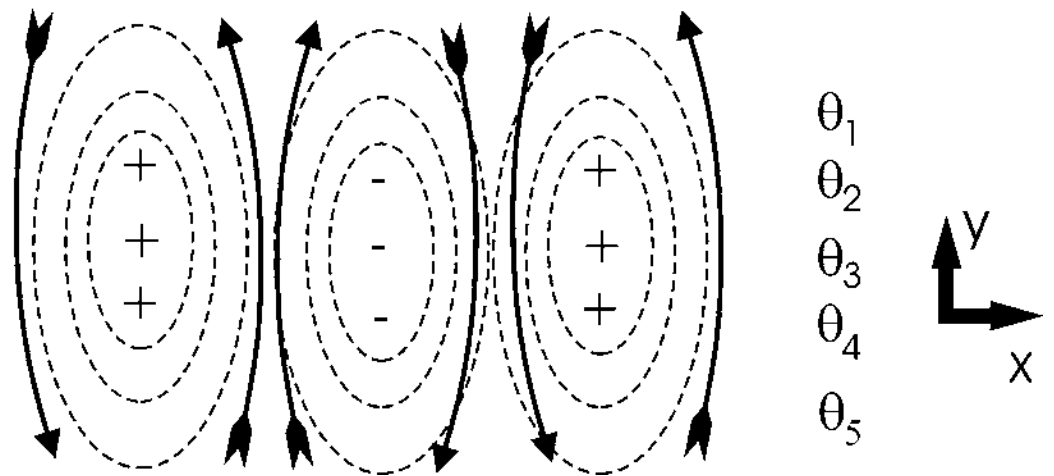
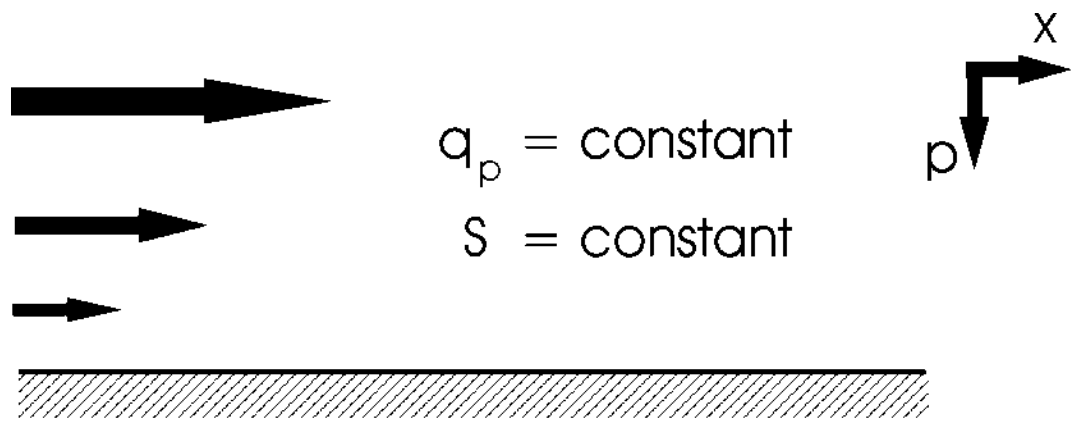


Figure 12.1

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