9. Quasi-geostrophic potential vorticity

The quasi-geostrophic system is at once more manageable and more intuitive if it is cast in the form of a potential vorticity conservation law and an invertibility principle. The potential vorticity conservation law can be obtained by combining a vorticity equation with the thermodynamic equation. The former can be obtained by taking the curl of (8.29), with the result

$$D_g \zeta_g + \beta v_g - f_0 \frac{\partial \omega}{\partial p} - \hat{k} \cdot \nabla \times \mathbf{F} = 0, \tag{9.1}$$

where

$$\zeta_g \equiv \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0} \nabla^2 \varphi + O(R_0) \tag{9.2}$$

is the *geostrophic relative vorticity.* Note that each of the terms in (9.1) is now of $O(R_0)$, and that to be consistent with this, we have replaced the variable f with a mean value in (9.1) and we also need to drop the $O(R_0)$ term in (9.2). The β term in (9.1) should also be replaced by a mean value. Thus to $O(R_0)$, (9.1) becomes the *quasi-geostrophic vorticity equation:*

$$D_g \nabla^2 \varphi + \beta_0 \frac{\partial \varphi}{\partial x} - f_0^2 \frac{\partial \omega}{\partial p} - \hat{k} f_0 \nabla \times \mathbf{F} = 0. \tag{9.3}$$

The thermodynamic equation, (8.30), is first cast in a slightly different form by substituting (8.33) and dividing through by $d\bar{\theta}/dp$, with the result

$$D_g \frac{1}{\mathcal{S}} \frac{\partial \varphi}{\partial p} + \omega = \frac{-\overline{\alpha} \dot{Q}}{\mathcal{S} \bar{\theta}} \tag{9.4}$$

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with

$$S \equiv \begin{cases} -\dfrac{\overline{\alpha}}{\overline{\theta}} \dfrac{d\overline{\theta}}{dp} & \text{atmosphere,} \\[2mm] -\dfrac{\overline{\alpha}}{\overline{\sigma}} \dfrac{d\overline{\sigma}}{dp} & \text{ocean,} \end{cases}$$

$$(9.5)$$

with $\overline{\alpha}$ given by the appropriate equation of state.

Note that (9.3) and (9.4) are functions of the variables φ and ω alone, given the distributions of Q and $\mathbf{F}$. We can form a predictive equation in the single variable φ by eliminating ω between the two equations. To do this, first take the derivative of (9.4) in p:

$$\frac{\partial}{\partial p} D_g \frac{1}{S} \frac{\partial \varphi}{\partial p} + \frac{\partial \omega}{\partial p} = -\frac{\partial}{\partial p} \frac{\overline{\alpha} Q}{S \overline{\theta}}. \tag{9.6}$$

Now note that expanding the left side of (9.6) gives

$$\begin{aligned} \frac{\partial}{\partial p} D_g \frac{1}{S} \frac{\partial \varphi}{\partial p} &= D_g \frac{\partial}{\partial p} \left(\frac{1}{S} \frac{\partial \varphi}{\partial p} \right) + \frac{\partial \mathbf{V}_g}{\partial p} \cdot \nabla \frac{1}{S} \frac{\partial \varphi}{\partial p} \\ &= D_g \frac{\partial}{\partial p} \left(\frac{1}{S} \frac{\partial \varphi}{\partial p} \right) + \frac{1}{S} \frac{\partial \mathbf{V}_g}{\partial p} \cdot \nabla \frac{\partial \varphi}{\partial p}. \end{aligned} \tag{9.7}$$

(The S can be taken outside because it is a function of p alone.) But since

$$\nabla \varphi = -f \hat{k} \times \mathbf{V}_g,$$

(9.7) can be written

$$\begin{aligned} \frac{\partial}{\partial p} D_g \frac{1}{S} \frac{\partial \varphi}{\partial p} &= D_g \frac{\partial}{\partial p} \left(\frac{1}{S} \frac{\partial \varphi}{\partial p} \right) - \frac{f}{S} \frac{\partial \mathbf{V}_g}{\partial p} \cdot \left(\hat{k} \times \frac{\partial \mathbf{V}_g}{\partial p} \right) \\ &= D_g \frac{\partial}{\partial p} \left(\frac{1}{S} \frac{\partial \varphi}{\partial p} \right). \end{aligned}$$

Thus (9.6) can be re-expressed as

$$D_g \frac{\partial}{\partial p} \left(\frac{1}{S} \frac{\partial \varphi}{\partial p} \right) + \frac{\partial \omega}{\partial p} = -\frac{\partial}{\partial p} \frac{\overline{\alpha} Q}{S \overline{\theta}}. \tag{9.8}$$

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Multiplying this by f_0, dividing (9.3) by f_0, and adding the result gives

$$D_g \left[\frac{1}{f_0} \nabla^2 \varphi + \frac{\partial}{\partial p} \left(\frac{f_0}{S} \frac{\partial \varphi}{\partial p} \right) \right] + \beta_0 v_g = \hat{k} \cdot \nabla \times \mathbf{F} - f_0 \frac{\partial}{\partial p} \frac{\overline{\alpha} Q}{S}. \qquad (9.9)$$

This can be written in a slightly different form by noting that $v_g = D_g y$:

$$D_g q_p = \hat{k} \cdot \nabla \times \mathbf{F} - f_0 \frac{\partial}{\partial p} \frac{\overline{\alpha} Q}{S}, \qquad (9.10)$$

where

$$q_p \equiv \frac{1}{f_0} \nabla^2 \varphi + \beta_0 y + \frac{\partial}{\partial p} \left(\frac{f_0}{S} \frac{\partial \varphi}{\partial p} \right) \qquad (9.11)$$

is the *pseudo potential vorticity.*

Note that, in contrast the Ertel's potential vorticity, q_p *is conserved following the geostrophic motion (or pressure surfaces).* Also note that *the invertibility relation (9.11) is a linear three-dimensional elliptic equation for φ.* Given certain boundary conditions, (9.10) and (9.11) constitute a closed system for φ, and the geostrophic wind and temperature (or density) perturbation can be recovered from (8.32) and (8.33).

Once again, there is a strong analogy with two-dimensional inviscid fluid dynamics, governed by

$$\frac{d\eta}{dt} = 0, \qquad (9.12)$$

$$\eta = \nabla^2 \psi, \qquad (9.13)$$

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where ψ is the streamfunction of the two-dimensional flow. The difference lies in two places: Whereas (9.13) is exact, (9.11) relies on the quasi-geostrophic approximation, and contains a *three-dimensional* elliptic operator rather than a two-dimensional operator.

The inversion of elliptic operators like (9.11) or (9.13) encompasses the principle of *action at a distance:* A localized distribution of vorticity, or potential vorticity, yields a more global distribution of wind and temperature. Solutions of both (9.11) and (9.13), because they are linear operators, are *linearly superposable.* One useful technique for carrying out the inversion is using the method of Green's functions. For a point vortex in a two-dimensional flow, the solution of

$$\nabla^2 \psi = A\delta(r),$$

where r is the radius from the source and A is the amplitude, is

$$\psi = -\frac{A}{2\pi} \ln r,$$

so that the tangential velocity, $\partial\psi/\partial r$, decays away from the point source as $1/r$. The action-at-a-distance principle is very analogous to the relationship between point changes and electric fields in electrostatics.

An analogous relation holds for the relationship between q_p and φ, as given by (9.11). To see this, let us first divide the q_p field according to

$$q_p = q_p' + \beta y,$$

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so that, from (9.11),

$$q_p' = \frac{1}{f_0}\nabla^2\varphi + \frac{\partial}{\partial p}\left(\frac{f_0}{\mathcal{S}}\frac{\partial\varphi}{\partial p}\right). \tag{9.14}$$

In the special case that $\mathcal{S}$ is constant (equal to $\mathcal{S}_0$), (9.14) becomes

$$q_p' = \frac{1}{f_0}\nabla^2\varphi + \frac{f_0}{\mathcal{S}}\frac{\partial^2\varphi}{\partial p^2}. \tag{9.15}$$

Now suppose we scale the horizontal distances in the system by

$$x, y \rightarrow \mathcal{S}_0^{1/2}f_0^{-1}\Delta p(x, y), \tag{9.16}$$

where Δp is some pressure scale, and scale pressure by Δp as well:

$$p \rightarrow (\Delta p)p. \tag{9.17}$$

Then (9.15), with the new independent variables, can be written

$$\frac{\mathcal{S}_0\Delta p^2}{f_0}q_p' = \nabla_3^2\varphi, \tag{9.18}$$

where the notation ∇_3^2 is used to indicate the three-dimensional Laplacian operator. A three-dimensional point potential vortex of amplitude $Af_0/(\mathcal{S}_0(\Delta p)^2)$ is associated with the geopotential distribution

$$\varphi = -\frac{A}{4\pi r}, \tag{9.19}$$

showing that the pressure distribution falls off inversely with radius from the source. The geostrophic velocities and temperatures fall off as $1/r^2$ in the horizontal and

vertical directions, respectively, and form dipole fields oriented horizontally and vertically, respectively.

Note that the inversion of both (9.11) and (9.13) results in streamfunction anomalies of the opposite sign of the vorticity anomalies (or of the opposite sign of q_p'/f_0, in the quasi-geostrophic case).

We now have in hand the core elements of a mode of thinking about the dynamics of quasi-balanced flows: the twin principles of potential vorticity conservation and invertibility. In the simplest balance approximation, quasi-geostrophy, the quantity that is conserved (to order Rossby number) is the pseudo potential vorticity, given by (9.11), and this is a linear elliptic function of the perturbation, φ, of the geopotential from its basic state value. Pseudo potential vorticity is conserved, according to (9.10), *following the geostrophic flow on pressure surfaces* (as opposed to the actual, three-dimensional flow).

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