2. Fundamental balance and conservations principles for quasi-balanced flow

A. Hydrostatic balance

We shall assume throughout the course that quasi-balanced flows are very nearly hydrostatic. In local Cartesian coordinates, the vertical momentum equation is

$$\frac{dw}{dt} = -\alpha\frac{\partial p}{\partial z} - g + 2\Omega u\cos\varphi + \frac{u^2 + v^2}{a} + F_z, \qquad (2.1)$$

where w is the vertical velocity, α is the specific volume, p is pressure, z is the upward vertical distance, g is the *effective* acceleration of gravity (which includes centripetal terms owing to the Earth's rotation), Ω is the angular velocity of the Earth's rotation, u is the zonal velocity, φ is the latitude, v is the meridional velocity, a is the mean radius of the earth and F_z is the vertical component of the acceleration owing to friction. The hydrostatic approximation is valid when the particle acceleration, Coriolis acceleration, and friction are all small compared to gravity:

$$\alpha\frac{\partial p}{\partial z} \simeq -g. \qquad (2.2)$$

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1. Application to the atmosphere

The atmosphere is well approximated by an ideal gas, whose specific volume is related to temperature, pressure, and water substance by

$$\alpha = \frac{R_d T_v}{p},$$ (2.3)

where R_d is the gas constant of dry air, p is the total pressure, and T_v is the *virtual temperature,* defined

$$T_v = T \left(\frac{1 + r/\epsilon}{1 + r_t} \right),$$ (2.4)

where T is the absolute temperature, r is the mass mixing ratio of water vapor, and r_t is the mass mixing ratio of all water substance. The mass mixing ratios are defined as the mass of substance per unit mass of air exclusive of all water substance. The total water mixing ratio, r_t, includes condensed as well as vapor-phase water. The quantity ϵ is the ratio of the molecular weight of water to the mean molecular weight of dry air and has a value of 0.622.

Substituting (2.3) into (2.2) gives

$$R_d T_v \frac{\partial \ln p}{\partial z} = -g.$$ (2.5)

Integrating this results in

$$p = p_0 \exp \left[\frac{-g_0 z_g}{R_d \overline{T}_v} \right],$$ (2.6)

where g_0 is a standard value of g and

$$z_g \equiv \frac{1}{g_0} \int_0^z g \, dz$$

is the *geopotential height.* In the troposphere, the fractional change of g with altitude is small and so z_g is nearly equal to z. *For the purposes of this course, we will always use z to mean geopotential height and take g as the acceleration of gravity at sea level.*

In (2.6), $\overline{T}_v$ is the mean virtual temperature, defined

$$\overline{T}_v \equiv \frac{1}{\ln \frac{p_0}{p}} \int_p^{p_0} T_v \frac{dp}{p}. \tag{2.7}$$

If the virtual temperature is constant with pressure, as is nearly true in the lower stratosphere, then (2.6) shows that pressure decreases exponentially with altitude.

In the atmosphere, it is common to use *pressure* as the *independent vertical coordinate,* rather than altitude. In this coordinate system, (2.2) is usually written

$$\frac{\partial \varphi}{\partial p} = -\alpha, \tag{2.8}$$

where φ is the *geopotential,* defined

$$\varphi \equiv \int_0^z g \, dz, \tag{2.9}$$

where g is in this instance the full effective gravitational acceleration. Using (2.3) and (2.7), (2.8) can be integrated to yield

$$\varphi = R_d \overline{T}_v \ln \frac{p_0}{p}. \tag{2.10}$$

Also, the physical distance between two fixed pressure surfaces, often referred to as the *thickness,* is

$$\Delta z_g = \frac{R_d \tilde{T}_v}{g} \ln \frac{p_2}{p_1}, \tag{2.11}$$

where in this case

$$\tilde{T}_v \equiv \frac{1}{\ln \frac{p_2}{p_1}} \int_{p_1}^{p_2} T_v \frac{dp}{p}. \qquad (2.12)$$

2. Application to the ocean

A convenient density variable to use in the ocean is σ, defined

$$\sigma \equiv (\rho - 1) \times 10^3, \qquad (2.13)$$

where ρ is the density in g cm^{-3}. In general, σ (or ρ) is a function of pressure, temperature, and salinity:

$$\sigma = \sigma(s, T, p).$$

The equation of state for sea water is not as simple as its atmospheric counterpart. It may be written approximately as

$$\alpha = \frac{c_1 + c_2 T + c_3 T^2 - c_4 S - c_5 TS}{p + c_6 + c_7 T - c_8 T^2 + c_9 S}, \qquad (2.14)$$

with α in cm^3 g^{-1}, p in bars, T in °C, and S in 0/00 (grams of dissolved substance per kilogram of sea water). The constants in (2.14) are

$$c_1 = 1752.73,$$

$$c_2 = 11.01,$$

$$c_3 = 0.0639,$$

$$c_4 = 3.9986,$$

$$c_5 = 0.0107$$

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$$c_6 = 5880.9,$$

$$c_7 = 37.592,$$

$$c_8 = 0.34395,$$

$$c_9 = 2.2524.$$

In the ocean, it is conventional to define z as *positive downward,* so the hydrostatic equation may be written

$$\frac{\partial p}{\partial z} = \rho g. \tag{2.15}$$

Since ρ is nearly constant, the vertical pressure gradient in the ocean is nearly equal to 1 db m^{-1}, where *db* stands for *decibar.*

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