1.204 Lecture 15

Dynamic programming: Knapsack

When multistage graphs don't work

- If the resource has many levels:
  - Large range of ints
  - Floating point number
- Then the multistage graph can't be constructed
  - And label correction is not a sufficient implementation for pruning
- We need a set representation instead
  - Different than our Set data structure, alas
- We keep all the elements in the solution at any stage in a set
  - We purge dominated elements
  - In a knapsack problem, for example, we purge any element whose weight is same or higher and its profit is same or lower than another element
  - This is how we implement pruning
- We still need to structure the problem so that feasibility constraints keep the size of the sets low

Knapsack problem

- **Problem is modeled as a series of decisions on whether to include item 1, item 2, item 3, …**
  - **Each item has a profit (benefit) and a weight (cost)**
  - **The knapsack has a maximum weight (cost)**
  - **Each project is either in or out of the knapsack**
    - **No fractional values allowed, as were in the greedy version**
- **Algorithm**
  - **Forward pass: builds sets instead of graph**
    - **Sets contain cumulative (profit, weight) pairs**
  - **Backward pass: traces sets back from sink to source to recover solution**
  - **Algorithm can produce solution for all weights less than or equal to maximum weight in a single run**

First example

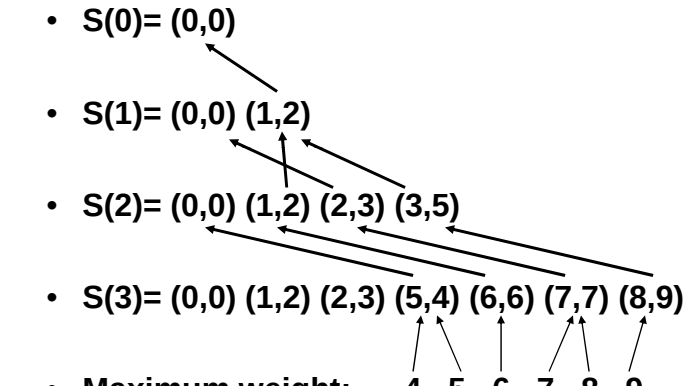
Item	Profit	Weight
0	0	0
1	1	2
2	2	3
3	5	4

- **Maximum weight= 9**
- **Item 0 is a sentinel with 0 weight, 0 profit always**

Forward pass: build sets

- S(0)= (0,0) S holds cumulative profit, weight
- S'= (1,2) S' is set of items to merge
- S(1)= (0,0) (1,2) S(n) is merged S(n-1) and S'
- S'= (2,3) (3,5)
- S(2)= (0,0) (1,2) (2,3) (3,5)
- S'= (5,4) (6,6) (7,7) (8,9)
- S(3)= (0,0) (1,2) (2,3) (5,4) (6,6) (7,7) (8,9)
  - Note that (3,5) is purged when S(3) is constructed
  - It is dominated by (5,4): higher profit, lower weight
- If maximum weight were 7, (8,9) pair would not be built
  - Infeasibility

Backward pass: get solution



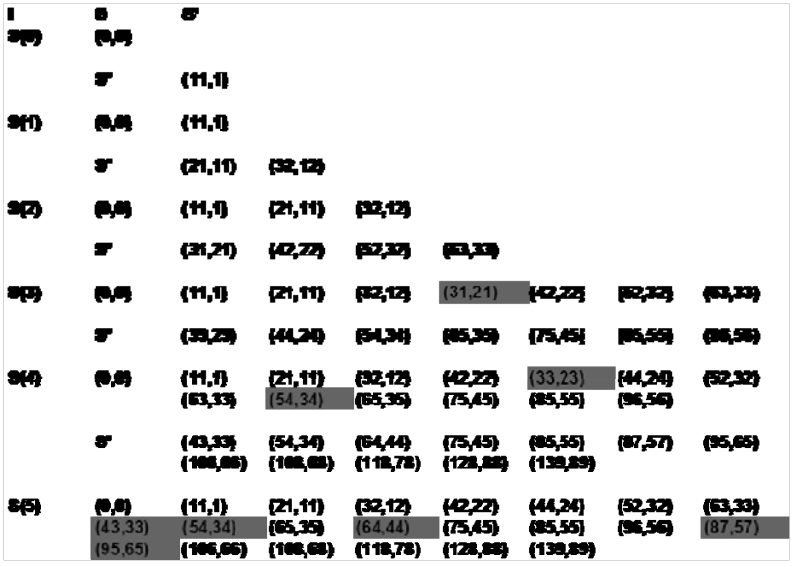
- Maximum weight:
  - Last pair is optimal (profit, weight) for entire problem
- If pair exists in previous set, item not in solution
- If pair not in previous set, item is in solution
  - Subtract item profit, weight and find that pair in previous set
  - Continue to trace back to source node

Second example

Item	Profit	Weight
0	0	0
1	11	1
2	21	11
3	31	21
4	33	23
5	43	33
6	53	43
7	55	45
8	65	55

- **Maximum weight 110**
- **Item 0 is sentinel**

Forward pass: build sets



Traceback uses same logic as before

Algorithm implementation

- **Follows examples, but there are complications:**
  - **We must keep each set S(i) to trace back the answer**
    - **In example 1, if we kept only the final set S(3), the pair (3,5) would have been purged and we would not be able to trace back the solution**
    - **Pairs dominated by pairs considered later can still be part of an optimal subsequence in the optimal solution**
    - **Storage requirements for all the sets are significant**
      - **We discard S' at each step**
  - **The sets are of varying and difficult-to-predict length**
    - **We use Java ArrayLists**
      - **O(1) add() method, which is all we use**
      - **Allow flexible number of pairs to be stored**
  - **The dominance operation is difficult to code**
  - **A sentinel, item 0, with 0 profit and 0 weight is needed**
    - **Must be at start of input regardless of input sort order**

Algorithm implementation 2

- **We sort the items in descending profit/weight order, as in the greedy algorithm**
  - **Putting 'good' items into the solution early usually allows more pruning to occur**
  - **Our dominance operation must handle any item order**
- **An alternative is to sort the items in descending weight order, if many items' weights are large relative to the knapsack maximum weight**
  - **This may make the sets smaller because feasibility constraints eliminate many combinations early**
- **It's always good to run the greedy version first**
  - **If it finds an integer solution, it's optimal**
  - **Even if it doesn't, its solution will give you insights on the nature of your problem data, and an approximate solution in case your DP doesn't terminate**

Generalizing the set-based dynamic programming code

- We use ints in this implementation
  - Can handle doubles but must use TOLERANCE when computing dominance to manage numerical error
- This implementation can be modified to handle other dynamic programming problems that can't be done with a multistage graph
  - E.g., the job scheduling dynamic program would keep a triplet (profit, time, deadline) instead of (profit, weight)
  - The dominance calculation would need to be modified to match the problem statement
    - The changes aren't as tough as writing it the first time

DPItem

```
public class DPItem implements Comparable {
   int profit;
   int weight;

   public DPItem(int p, int w) {
        profit= p;
        weight= w;
   }
   public boolean equals(Object other) {
        DPItem o= (DPItem) other;
        if (profit == o.profit && weight == o.weight)
                return true;
        else
                return false;
   }
   public int compareTo(Object o) {
        DPItem other = (DPItem) o;
        double ratio= (double) profit/weight;
        double otherRatio= (double) other.profit/other.weight;
        if (ratio > otherRatio)       // Descending sort
                return -1;
        else if (ratio < otherRatio)
                return 1;
        else
                return 0;
}  }            // toString() method not shown
```

DPSet constructor, extend()

```
public class DPSet {
   ArrayList<DPItem> data;  // Flexible capacity, fast add
   private static int capacity;           // Maximum weight

   public DPSet() {
        data= new ArrayList<DPItem>();
   }

   public static void setCapacity(int c) {
        capacity= c;
   }

   public DPSet extend(DPItem other) {     // Add item to set
        DPSet result= new DPSet();
        for (DPItem i: data) {
               int cumWgt= i.weight + other.weight;
               if (cumWgt <= capacity) {
                 int cumProf= i.profit + other.profit;
                 result.data.add(new DPItem(cumProf, cumWgt));
               }
        }
        return result;
   }
```

DPSet merge(), p. 1

```
public DPSet merge(DPSet other) {
   // Merges DPSet other with this DPSet, with dominance pruning

   // Items in any input sort order wind up in weight order
   DPSet result= new DPSet();
   // Define limits for while loop on DPSet other
   int indexOther= 0;
   int maxIndexOther= other.data.size()-1;
   // Last item profit used for dominance check at end of set
   int lastItemProfitOther= other.data.get(maxIndexOther).profit;

   // Define limits for while loop on this DPSet
   int index= 0;
   int maxIndex= data.size()-1;
   int lastItemProfit= data.get(maxIndex).profit;

   // Continues on next slide, which compares items and other items
```

Dominance

- If item weight < other weight
  - Write item to results; it cannot be dominated
  - If other profit <= item profit, other is dominated; skip it
    - Keep looping over next other items ‘til not dominated
- If item weight= other weight
  - If item profit >= other profit
    - Skip other item; it’s dominated
  - Else skip item; it’s dominated
  - Don’t write either of them into solution yet
    - Either may be dominated by a previous pair.
    - Wait for next comparison
- If other weight < item weight
  - Same logic as first case holds

DPSet merge(), p. 2

```
while (index <= maxIndex || indexOther <= maxIndexOther) {
  if (index <= maxIndex && indexOther <= maxIndexOther) { // Both ok
    DPItem item= data.get(index);
    DPItem otherItem= other.data.get(indexOther);
    if (item.weight < otherItem.weight) {
      result.data.add(item);  // Add item; not dominated by other item
      index++;
      while (otherItem.profit< item.profit && indexOther< maxIndexOther)
        otherItem= other.data.get(++indexOther); // Other dominated,skip
    } else if (item.weight == otherItem.weight) {
        if (item.profit >= otherItem.profit) // Other item dominated
          indexOther++;
        else
          index++;                          // Item dominated
    } else {   // otherItem.weight < item.weight
      result.data.add(otherItem);  // Add other item, not dominated
      indexOther++;
      while (item.profit < otherItem.profit && index < maxIndex)
        item= data.get(++index); // Item dominated; skip it
    }
  }    //  Continues on next slide, within while loop; end condition
```

DPSet merge(), p. 3

```
    // One loop index is already at end. Handle remaining in other set

    else if (index > maxIndex) { // Only other items left to consider
      while (indexOther <= maxIndexOther) {
        DPItem otherItem= other.data.get(indexOther);
        if (otherItem.profit > lastItemProfit)
          result.data.add(otherItem);
        indexOther++;
      }
    } else {   // indexOther > maxIndexOther. Only items left
      while (index <= maxIndex) {
        DPItem item= data.get(index);
        if (item.profit > lastItemProfitOther)
          result.data.add(item);
        index++;
      }
    }
  }
  return result;
}
```

DPKnap constructor, knapsack()

```
public class DPKnap {
   private DPItem[] items;    // Input items
   private int m;             // Capacity of knapsack
   private DPSet[] sets;      // Subsequences, sets
   private DPItem[] solution; // Solution with optimal items only

   public DPKnap(DPItem[] i, int maxCap) {
        items= i;
        m= maxCap;
        sets= new DPSet[items.length];
        solution= new DPItem[items.length];
   }

   public void knapsack() {
        buildSets();
        backPath();
        outputSolution();
   }
```

DPKnap buildSets()

```
private void buildSets() {
     DPSet.setCapacity(m);
     // Build set 0 with node 0
     DPSet s= new DPSet();
     // Add item 0 to set 0. Sentinel w/0 profit, weight.
     s.data.add(items[0]);
     sets[0]= s;

     // For sets 1 and above
     for (int i= 1; i < sets.length; i++) {
            // Add item and find cumulative profit, weight pairs
            DPSet sNext= s.extend(items[i]);
            // Merge, with dominance, with prior set
            s= s.merge(sNext);
            // Store new set; needed to trace back solution
            sets[i]= s;
     }
}
```

DPKnap backPath() 1

```
private void backPath() {
   int lastSetIndex= sets.length-1;   // Start at last set
   int lastSetItem= sets[lastSetIndex].data.size()-1;
   DPItem lastItem= sets[lastSetIndex].data.get(lastSetItem);

   int cumProfit= lastItem.profit;
   int cumWeight= lastItem.weight;
   DPItem prevItem= lastItem;

   for (int i= lastSetIndex-1; i >= 0; i--) {
     boolean itemFound= false; // Is item in previous set
     int prevSetIndex= i+1;
     DPSet currSet= sets[i];
     int currItemIndex= currSet.data.size()-1;
     for (int j= currItemIndex; j >= 0; j--) {
       DPItem currItem= currSet.data.get(j);
       if (currItem.equals(prevItem)) {
         itemFound= true;
         break;
       }
       if (currItem.weight < prevItem.weight)
         break; // No need to search further
     }         // Continued on next slide
```

DPKnap backPath() 2

```
      // Pair not found in preceding set; item is in solution
      if (!itemFound) {
        solution[prevSetIndex]= items[prevSetIndex];
        cumProfit -= items[prevSetIndex].profit;
        cumWeight -= items[prevSetIndex].weight;
        prevItem= new DPItem(cumProfit, cumWeight);
      }  // else keep searching for prev item in the next set
    }
  }
```

DPKnap outputSolution()

```
  private void outputSolution() {
      int totalProfit= 0;
      int totalWeight= 0;
      System.out.println("Items in solution:");
      // Position 0 in solution is sentinel; don't output
      for (int i= 1; i < solution.length; i++)
             if (solution[i] != null) {
                    System.out.println(items[i]);
                    totalProfit += items[i].profit;
                    totalWeight += items[i].weight;
             }
      System.out.println("\nProfit: " + totalProfit);
      System.out.println("Weight: " + totalWeight);

  }
```

DPKnap main()

```
public static void main(String[] args) {
   // Sentinel- must be in 0 position even after sort
   DPItem[] list= {new DPItem(0, 0),
               new DPItem(11, 1),
               new DPItem(21, 11),
               new DPItem(31, 21),
               new DPItem(33, 23),
               new DPItem(43, 33),
               new DPItem(53, 43),
               new DPItem(55, 45),
               new DPItem(65, 55),
   };
   Arrays.sort(list, 1, list.length); // Leave sentinel in position 0
   int capacity= 110;
   // Assume all item weights <= capacity. Not checked. Discard such
   items.
   // Assume all item profits > 0. Not checked.  Discard such items.
   DPKnap knap= new DPKnap(list, capacity);
   knap.knapsack();
}
```

DPKnap example 2 output

```
Items in solution:
Profit: 11 weight: 1
Profit: 21 weight: 11
Profit: 31 weight: 21
Profit: 43 weight: 33
Profit: 53 weight: 43

Profit: 159
Weight: 109
```

Problem size

- **How large a problem will the set-based dynamic programming approach solve?**
  - **It's highly data-dependent**
  - **If you're lucky, you may solve a problem with hundreds or even thousands of items**
    - **If maximum capacity is low, so feasibility check cuts out many combinations**
    - **If profit/weight sort or other heuristic is effective in pruning many combinations from the sets**
  - **If you're unlucky, the program will get to about 40 or 50 items and stall (2^{40} is a large number)**
    - **You may run out of storage for the sets before your computation time also becomes excessive**

Dynamic programming

- **Generally used on smaller 0-1 decision problems, often of size 20 to 40, or perhaps 100 items**
  - **Dynamic programming occasionally works on large problems**
- **Generally used on 'integrated problems' that don't decompose into a master problem and subproblems**
  - **We will study branch-and-bound methods next, which are better suited for problems that decompose**
- **With multistage graphs, dynamic programming is a label correcting shortest path algorithm on a graph (that we don't actually need to build)**
  - **One source (origin), one sink (destination**
  - **Running time depends on the size of the virtual graph**
- **With sets, dynamic programming uses a dominance criterion**
  - **Not as efficient as label correction, but a graph can't be built**
  - **More effective pruning by comparing all states in a stage**
- **Keys are to use pruning/dominance and feasibility constraints to keep the graph or set sizes small**
  - **Efficient implementations that don't store unnecessary data or do unnecessary calculations can help significantly**

1.204 Computer Algorithms in Systems Engineering

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