# QUIZ 1
Solutions

Problem 1. Patrolling Police Car.

A patrolling police car is assigned to the rectangular sector shown in the figure. The sector is bounded on all four sides by a roadway that requires 50% of the police car's patrolling time. The other 50% of the time, the car patrols the inner rectangular part of the sector. Thus, at a random time when the police car is available for dispatch, the police car's location is equally likely to be drawn from a uniform distribution over the bounding roadway or by a uniform distribution over the rectangular part of the sector inside the bounding roadway. Travel is right-angle or the Manhattan metric, with directions of travel parallel to the sides of the rectangle. 911 calls for service are also distributed randomly over the rectangular sector, in the same way as the police car and independently of the location of the police car. That is, 50% of the 911 calls are uniformly distributed over the bounding roadway and 50% uniformly distributed over the inner rectangular part of the sector. Given a random call for service at a give location, the police car will follow a minimum distance right-angle path from its current location to the location of the call. Thus, we assume that the police car can exit the bounding
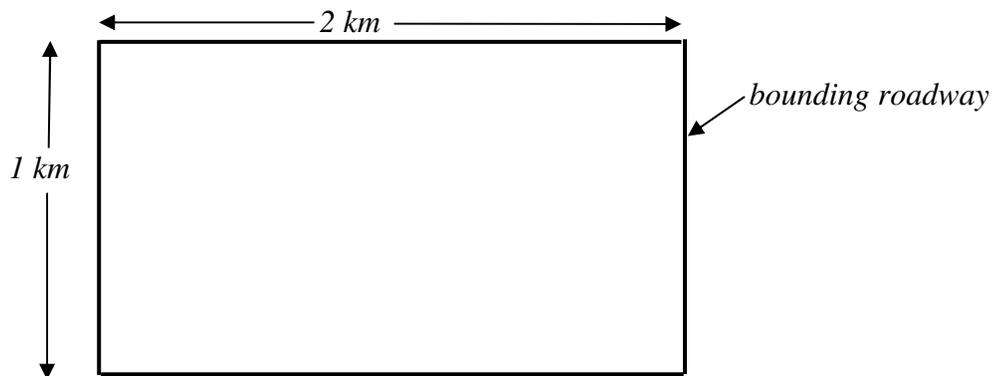
roadway at any point, and – as is usual with the right angle metric, we ignore the complication of 'city blocks', assuming instead an infinitely divisible right-angle travel space within the region.

> (a) *Find the mean distance traveled by the police car in response to a random call for service.*

    Key methods: - "Divide & Conquer"
                      - Expected distance between two random points

The police car can either be on the highway or in the inner part of the sector, and it is the same for the random call for service. Therefore, we have four different cases. The probabilities of those events are summarized in the table:
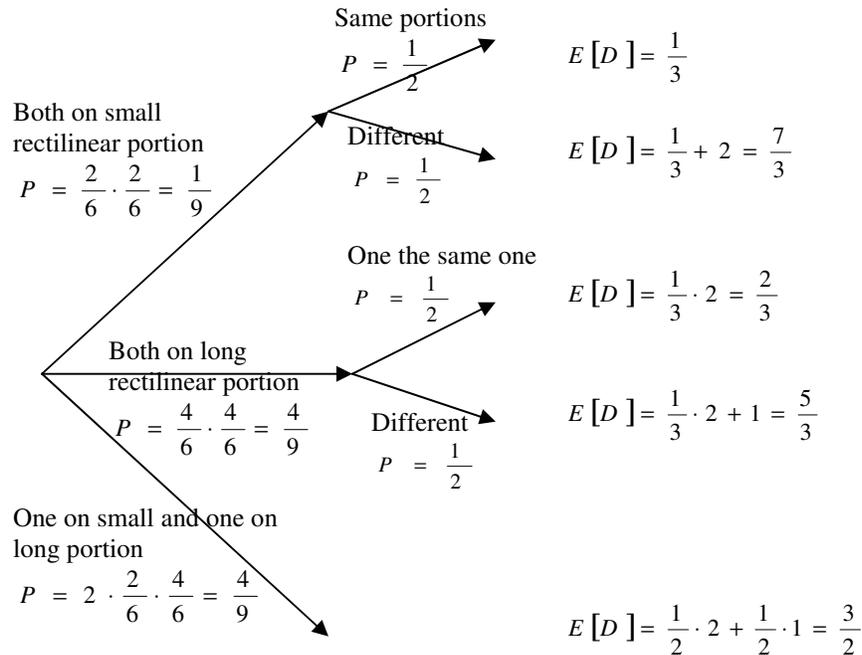
		Police Car	
		Highway	Inner Part
911 Call	Highway	Case 1: 1/4	Case 2: 1/4
911 Call	Inner Part	Case 3: 1/4	Case 4: 1/4

For each case, we will determine the expected distance.

- Case 1:
  We have to use the method "Divide and Conquer" again. If the call and the car are on two opposite portions of the highway, then the car has to cross the inner rectangular region, whereas if they are on the same portion of the highway, the car does not have to travel across that region. Thus, the expected distance depends on which part of the highway the call and cars are.
  Note: the car can go off the highway, therefore, the problem cannot be reduced to a 1-D problem.

Both on small rectilinear portion
$$P = \frac{2}{6} \cdot \frac{2}{6} = \frac{1}{9}$$

Same portions
$$P = \frac{1}{2}$$
$$E[D] = \frac{1}{3}$$

Different
$$P = \frac{1}{2}$$
$$E[D] = \frac{1}{3} + 2 = \frac{7}{3}$$

Both on long rectilinear portion
$$P = \frac{4}{6} \cdot \frac{4}{6} = \frac{4}{9}$$

One the same one
$$P = \frac{1}{2}$$
$$E[D] = \frac{1}{3} \cdot 2 = \frac{2}{3}$$

Different
$$P = \frac{1}{2}$$
$$E[D] = \frac{1}{3} \cdot 2 + 1 = \frac{5}{3}$$

One on small and one on long portion
$$P = 2 \cdot \frac{2}{6} \cdot \frac{4}{6} = \frac{4}{9}$$

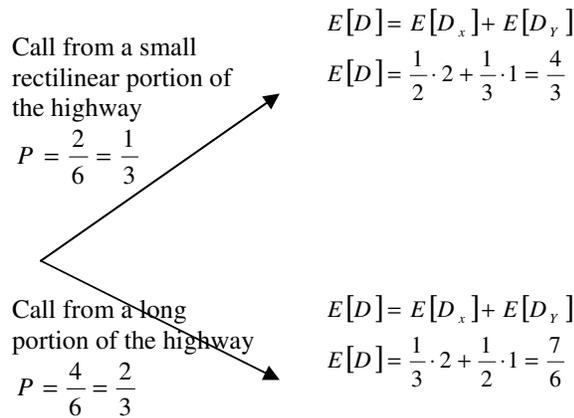$$E[D] = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 1 = \frac{3}{2}$$

Therefore, for the expected distance when they are both on the highway is:

$$E[D] = \frac{1}{9} \cdot \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{9} \cdot \frac{1}{2} \cdot \frac{7}{3} + \frac{4}{9} \cdot \frac{1}{2} \cdot \frac{2}{3} + \frac{4}{9} \cdot \frac{1}{2} \cdot \frac{5}{3} + \frac{4}{9} \cdot \frac{3}{2} = \frac{4}{3}$$

- Case 2:
  The call comes from the highway and the police car is in the inner part of the sector. The distance will depend on which portion of the highway the call originates from. The police car will have to either all the way up or down, or right or left.

Call from a small
rectilinear portion of
the highway

$$P = \frac{2}{6} = \frac{1}{3}$$

$$E[D] = E[D_x] + E[D_Y]$$
$$E[D] = \frac{1}{2} \cdot 2 + \frac{1}{3} \cdot 1 = \frac{4}{3}$$

Call from a long
portion of the highway

$$P = \frac{4}{6} = \frac{2}{3}$$

$$E[D] = E[D_x] + E[D_Y]$$
$$E[D] = \frac{1}{3} \cdot 2 + \frac{1}{2} \cdot 1 = \frac{7}{6}$$

Therefore, the expected distance is this case is $E[D] = \frac{1}{3} \cdot \frac{4}{3} + \frac{2}{3} \cdot \frac{7}{6} = \frac{11}{9}$

- Case 3:
  It is similar to Case 2, with the roles for car and the call inverted.

  Thus, the expected distance is $E[D] = \frac{11}{9}$

- Case 4:
  This is the easiest case. They are both from the uniform region, therefore the

  answer is simply: $E[D] = E[D_X] + E[D_Y] = \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot 1 = 1$

Finally, the expected distance is $\boxed{E[D] = \frac{1}{4}\left(1 + 2 \cdot \frac{11}{9} + \frac{4}{3}\right) = \frac{43}{36}}$.

(b) Now assume we have a barrier to travel that rises vertically from the midway point on the lower boundary of the sector to 1/2 km inside the sector. (See figure on next page.) The barrier does not stop travel along the bounding roadway near the barrier. It only prohibits east-west and west-east travel through the barrier

within the interior of the sector. Now find the mean distance traveled in the presence of the barrier.

There are a lot of cases to consider when computing the expected distance. In order to reduce the computations we have to make, we will use the "**perturbation" method**:
$$E[D] = E[D_{normal}] + E[D_{perturbation}].$$
The barrier creates a perturbation when the call and the police car are on both sides of it. An example is given on the figure below:
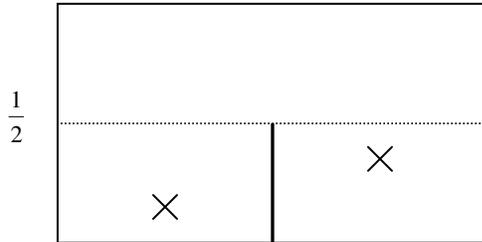
The police car might go round the barrier from the top or the bottom.
Thus, we can define four areas of interest:
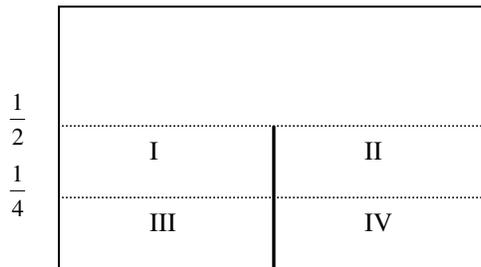
- The cases were the car and the call are in I and II, or III and IV are similar. Let's consider the case where they are in III and IV, then the extra travel distance is $D_{perturbation} = 2\min[Y_1, Y_2]$. Therefore, the expected perturbation distance is:

$$E[D_{perturbation}] = 2 \cdot \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{6}.$$

- The cases were the car and the call are in I and IV, or II and III are similar. Let's consider the case where they are in I and IV. Then the extra travel distance is $D_{perturbation} = 2 \cdot \min\left[\frac{1}{2} - Y_I, Y_{IV}\right]$. Since Y_I is uniformly distributed over [0.25; 0.5], $\frac{1}{2} - Y_I$ is uniformly distributed over [0; 0.25]. Thus, $E[D_{perturbation}] = \frac{1}{6}$ as well.

Therefore, the expected value of the extra distance is:

$$E[D_{perturbation}] = \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{6}$$

Now, the total expected distance is: $E[D] = \frac{43}{36} + \frac{1}{8} \cdot \frac{1}{6} = \frac{175}{144} \approx 1.25$

Problem 2. Halloween Treat.

Halloween pumpkins are planted in parallel straight-line rows on a very big field on a farm in New England. The field is so big that we ignore any boundary effects. The parallel rows are two meters apart. Research has shown that the planter of pumpkin seeds was a sort of random fellow, and that we can accurately model the spatial distribution of pumpkins along any row as a homogeneous spatial Poisson process with parameter $\gamma = 1$ pumpkin per meter. Any given pumpkin has one chance in ten of becoming a giant pumpkin for Halloween. A giant pumpkin weighs more than 30 pounds! Otherwise it is a regular sized pumpkin, weighing less than or equal to 30 pounds.   The sizes of pumpkins are mutually independent.

(a) If we walk along any given row of pumpkins, what is the probability that in a walk of 100 meters we will have passed 3 or more giant pumpkins on that row?

The giant pumpkins can also be modeled as a homogeneous spatial Poisson process, with parameter $v = \dfrac{\gamma}{10} = 0.1$ pumpkin per meter.

Therefore, the probability of passing 3 or more giant pumpkins in a walk of 100 meters is
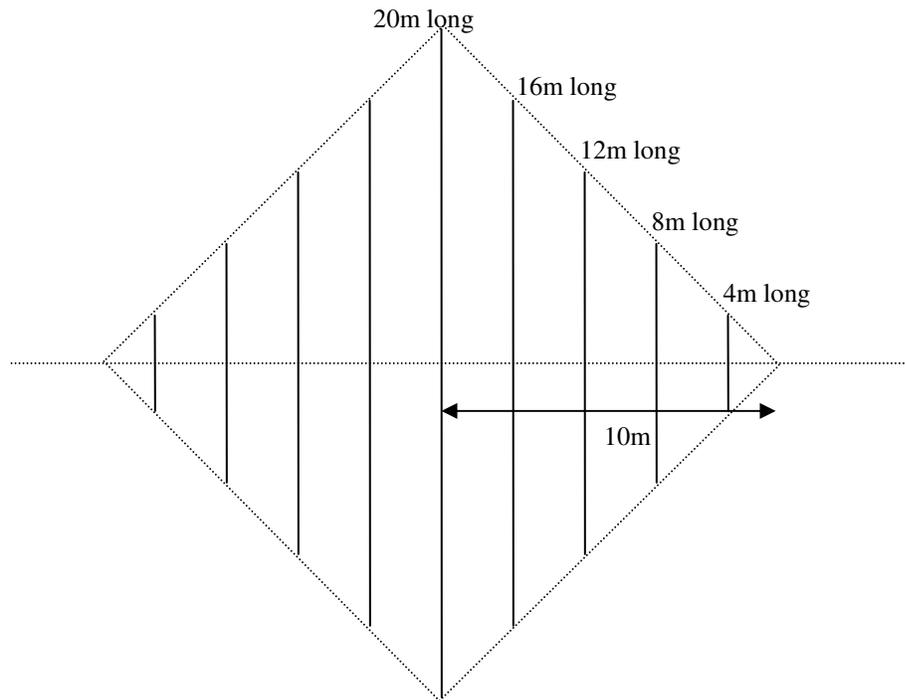
$P = P(k \ in \ 100m \geq 3)$

$P = 1 - (P(0 \ in \ 100m) + P(1 \ in \ 100m) + P(2 \ in \ 100m))$

$P = 1 - (\sum_{k=0}^{2} \frac{(100v)^k e^{-100v}}{k!})$

$P \approx 0.997$

(b) If we are standing at the location of a random giant pumpkin, what is the probability that at least 3 additional giant pumpkins are within a right-angle distance of 10 meters from our location? (This question includes pumpkins in

near-by parallel rows.)

20m long

16m long

12m long

8m long

4m long

10m

We are considering a total of 100 meters.
Therefore, because we are considering a Poisson process, we can use the result to the previous questions where we were also considering a total of 100 meters: $P \approx 0.997$

Problem 3

Consider the operation of a government office with a single clerk. For all practical purposes, the office constitutes a queueing system with infinite queue capacity (there are lots of chairs to accommodate all those waiting for service). This operation can be modeled as a M/M/1 system with one complication. The clerk, a capricious person, keeps a record of the time when every person seeking service arrives at the office. Whenever there are two or more people waiting at the time when a service is completed, the clerk selects the next person to be served in the following way: with probability 0.6 she selects the person with the earliest arrival time (i.e., the person who has been waiting the longest) and with probability 0.4 the one with the latest arrival time (i.e., the one who arrived most recently). The arrival rate at the office, λ, is equal to 4 per hour and the service rate, μ, is 6 per hour.

(a) Please draw the state transition diagram for this queueing system, with the state of the system indicating the number of people in the office (waiting or being served).

The state transition diagram of this system is:
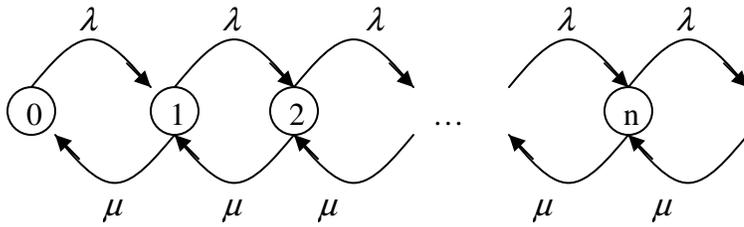
(b) Consider the expected values L, W and B (respectively, the average number of persons at the office, the average total time spent there, and the average length of a busy period for the clerk). Which, if any, of these three quantities for this queueing system is different from the corresponding quantities for a first-come, first-served (FCFS) M/M/1 system? Please explain briefly.

None of the expected values L, W or B changes. To derive those expected values, we never had to consider the type of queueing discipline used. Therefore, the expressions are independent of the queueing discipline.

(c) How would the variance of W compare between this office and the FCFS M/M/1 system? Please explain briefly.

However, the variance is not independent of the queueing discipline. On a FCFS system, one knows that in order to be served, all the people in front in the queue have to be served first. In our system, being the nth person in the queue does not mean that we have to wait for n-1 people to be served first. If there is nobody behind in the queue, the nth person can be the next person to be served, or if there is someone, that person could go first and then all the other people in front in the queue. Therefore, the variance is greater in our system than in the FCFS M/M/1 system.

Numerical answers should be provided to questions (d) – (f) with brief explanations.

(d) Person A arrives to find exactly four other persons present at the office, including the one being served (but not counting the clerk). What is the probability that A will be the next person to be served?

If A is the next to be served, then it means that the clerk chose to serve the person with the *latest* arrival and that A is that person. A is the person with the latest arrival in the queue if no other persons arrive before the completion of the ongoing service.

That probability is the probability that the event following A's arrival is the service completion rather than an arrival. It is given by:

$$\frac{P(service\ completition\ in\ time\ iterval\ \Delta t)}{P(service\ completition\ or\ arrival\ in\ time\ iterval\ \Delta t)} = \frac{\mu \cdot \Delta t}{(\mu + \lambda) \cdot \Delta t} = \frac{\mu}{\mu + \lambda} \ .$$

Thus, the probability that A will be the next person to be served is

$$0.4 \cdot \frac{\mu}{\mu + \lambda} = 0.4 \frac{6}{4 + 6} = 0.24 \ .$$

(e) Person B arrives to find only one other person at the office, i.e., the person being currently served by the clerk. What is the probability that B will be the next person to be served by the clerk?

B will be the next person to be served by the clerk if no one else arrives, or if other people arrive but the clerk chooses to serve the person with the *earliest* arrival time. Therefore, the probability is of B being the next one to be served is:

$$P = P(no\ other\ arrival) + [1 - P(no\ other\ arrival)] \cdot 0.6$$

$$P = \frac{\mu}{\mu + \lambda} + 0.6(1 - \frac{\mu}{\mu + \lambda})$$

$$P = \frac{21}{25} \approx 0.84$$

(f) For a randomly chosen person, C, who enters this office (and assuming steady state conditions), what is the probability that he will be the next customer served? [HINT: Consider, all the possible states that C may find the office in, and think of what will happen for each state.]

If there are no customers in the system, C is the next one to be served with probability 1. If there is one person, C is the next one to be served if:

- nobody else arrives, which has a probability of $\frac{\mu}{\mu + \lambda}$

- other people arrive, but the clerk chooses to serve the person with the earliest arrival time, which has a probability of $0.6 \cdot (1 - \frac{\mu}{\mu + \lambda})$

If there are two people or more in the system, C is the next one to be served if the clerk chooses to serve the person with the latest arrival time and nobody arrives between C's arrival and the completion of the actual service. The associated probability is $0.4 \frac{\mu}{\lambda + \mu}$.

Thus, the probability for a random person C to be the next person to be served is

$$P = P_0 + P_1 \cdot [\frac{\mu}{\lambda + \mu} + 0.6 \cdot (1 - \frac{\mu}{\lambda + \mu})] + (1 - P_0 - P_1) \cdot 0.4 \cdot \frac{\mu}{\lambda + \mu}$$

$$P = \frac{47}{75}$$