Problem Set 6

Due: December 6, 2006

Problem 1

(a)

$$\begin{aligned}\sum_{i\in N} P_i &= \sum_{i\in N} \{(\text{indegree of } i) - (\text{outdegree of } i)\} \\ &= \sum_{i\in N}(\text{indegree of } i) - \sum_{i\in N}(\text{outdegree of } i) \\ &= \sum_{i\in N}\sum_{(k,i)\in A} 1 - \sum_{i\in N}\sum_{(i,k)\in A} 1 = 0\,.\end{aligned}$$

Both of the last two sums count every directed arc of the network exactly once: the left-hand sum from the point of view of the tails and the right-hand sum from the point of view of the heads. Hence the difference of the two sums is zero (note that every arc (i,j) contributes exactly one to the outdegree of i and one to the indegree of j).

(b) In order to have a directed Euler tour, we must have $P_i' = 0$ for all nodes. Parallel to the undirected version, we add artificial arcs (i,j) between supply nodes $i \in S$ and demand nodes $j \in D$. Unlike the undirected version, where one additional arc was sufficient to make any odd node even, here it may be necessary to add many arcs to a node whose $|P_i|$ is large. In order to minimize the total length of arcs added, we construct $\sum_{i\in S} P_i$ minimum distance paths between the supply nodes and demand nodes. In order to ensure $P_i' = 0$ for all nodes, we require $\sum_{j\in D} x_{ij} = P_i,\ \forall i \in S$, which implies that

$$P_i' = P_i - \text{outdegree of new artificial arcs} = P_i - \sum_{j\in D} x_{ij} = 0\,.$$

Similarly, we require $\sum_{i\in S} x_{ij} = -P_j,\ \forall j \in D$, which ensures that

$$P_j' = P_j + \text{indegree of new artificial arcs} = P_j + \sum_{i\in S} x_{ij} = 0\,.$$

Here, x_{ij} represents the number of new artificial paths between nodes i and j. Since we now have $P_i' = 0,\ i \in S$ and $P_j' = 0,\ j \in D$, we can construct an Euler tour. It is certainly possible to use a link more than twice (See, for example, arcs (d,e) and (b,a) in the next part).

(c)

Step 1: $S = \{b,d,g\}$ with $P_b = P_d = P_g = 1$, and $D = \{a,e\}$ with $P_a = -2, P_e = -1$. By inspection,

$$\begin{aligned} d(b,a) &= 5, & d(b,e) &= 17 \\ d(d,a) &= 14, & d(d,e) &= 3 \\ d(g,a) &= 20, & d(g,e) &= 9 \end{aligned}$$

Step 2:

$$\begin{aligned}
\text{minimize} \quad & z = 5x_{ba} + 17x_{be} + 14x_{da} + 3x_{de} + 20x_{ga} + 9x_{ge} \\
\text{subject to} \quad & x_{ba} + x_{be} = 1 \\
& x_{da} + x_{de} = 1 \\
& x_{ga} + x_{ge} = 1 \\
& x_{ba} + x_{da} + x_{ga} = 2 \\
& x_{be} + x_{de} + x_{ge} = 1 \\
& x_{ij} \in \{0, 1, 2, \dots\}
\end{aligned}$$

There are only three feasible integer solutions to this problem, so we enumerate these and check the value of the objective function in each case:

(1) $x_{ba} = 1, x_{be} = 0, x_{da} = 1, x_{de} = 0, x_{ga} = 0, x_{ge} = 1: \; z = 5 + 14 + 9 = 28$

(2) $x_{ba} = 0, x_{be} = 1, x_{da} = 1, x_{de} = 0, x_{ga} = 1, x_{ge} = 0: \; z = 17 + 14 + 20 = 51$

(3) $x_{ba} = 1, x_{be} = 0, x_{da} = 0, x_{de} = 1, x_{ga} = 1, x_{ge} = 0: \; z = 5 + 3 + 20 = 28$

Both solutions 1 and 3 are optimal. We choose the solution 1.

Step 3: We add paths from $b \to a$, $d \to a$ ($d \to e \to b \to a$), and $g \to e$ ($g \to d \to e$). See Figure 2.

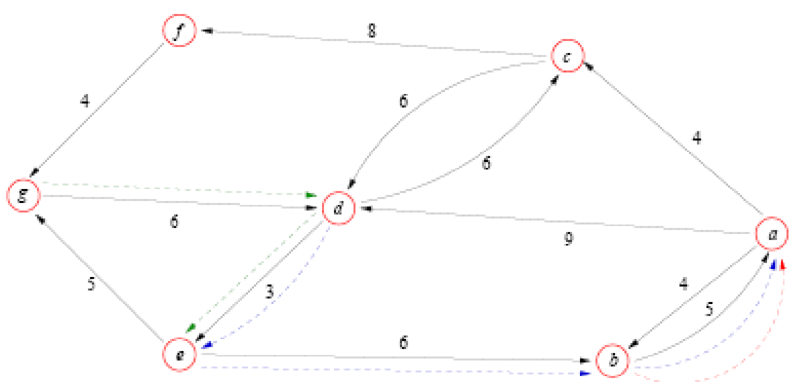

Figure 2: Step 3 of Problem 6.6(c)

Step 4: $b \to a \to c \to d \to c \to f \to g \to d \to e \to g \to d \to e \to b \to a \to d \to e \to b \to a \to b$ is one possible tour.

(d) The suggested method forces us to traverse every undirected arc twice (once in each direction), which may not be optimal.

Problem 2

Let G be the graph under consideration with node set N. Let x be some point between nodes p and q. For all $x' \in G$, x is reachable from x' only through p or q. Therefore, the shortest path from x to x' is through either p or q. As a function of x, $d(x, x')$ is a piecewise linear function with slope ± 1 at each value of x. Similarly, it is easy to see that $d(x,p)$ and $d(x,q)$, as functions of x are linear with slope ± 1. Since $m(x) = \max_{j \in N} d(x,j)$ by definition, then $m(x)$ is the maximum of piecewise linear functions each having slope ± 1 for each value of x. So, $m(x)$ itself is also a piecewise linear function of x with slope at each point equal to ± 1.

Let $d_{(p,q)}(p,x)$ and $d_{(p,q)}(q,x)$ be the distances, along link (p,q) from x to p and from x to q, respectively. Because the slope of $m(x)$ is ± 1 for each point x on the link (p,q), as we move from $z = p$ to $z = x$, $m(z)$ can decrease by no more than $d_{(p,q)}(p,x)$. Similarly, as we move from $z = x$ to $z = q$, $m(z)$ can increase by no more than $d_{(p,q)}(x,q)$. That is,

$$\begin{aligned} m(x) &\geq \max\left[m(p) - d_{(p,q)}(p,x),\ m(q) - d_{(p,q)}(x,q)\right] \\ &\geq \frac{1}{2}\left[m(p) - d_{(p,q)}(p,x)\right] + \frac{1}{2}\left[m(q) - d_{(p,q)}(x,q)\right] \\ &= \frac{m(p) + m(q) - \ell(p,q)}{2} \end{aligned}$$

The second inequality follows since the maximum of two values is no smaller than their average.

Problem 3

(a) As suggested, we can prove the results by contraposition. Let's suppose that the set of nodes T contains no solution to the 1-median problem. Then, the solution one of S's nodes. Let $y \in S$ be that node.

Then

$$\begin{aligned} J(y) &= \sum_{j \in T} h(j) d(y,j) + \sum_{j \in S} h(j) d(y,j) \\ &= \sum_{j \in T} h(j)[d(y,t) + d(t,j)] + \sum_{j \in S} h(j) d(y,j) \\ &= H(T) d(y,t) + \sum_{j \in T} h(j) d(t,j) + \sum_{j \in S} h(j) d(y,j) \\ &\geq H(S) d(y,t) + \sum_{j \in T} h(j) d(t,j) + \sum_{j \in S} h(j) d(y,j) \\ &= \sum_{j \in T} h(j) d(t,j) + \sum_{j \in S} h(j)[d(y,j) + d(y,t)] \\ &= \sum_{j \in T} h(j) d(t,j) + \sum_{j \in S} h(j) d(t,j) \\ &= J(t) \end{aligned}$$

This implies that the 1-median could be located at $t \in T$, which contradicts our initial assumption.

Therefore, the set of nodes T contains at least one solution to the 1-median problem on G(N,A).

(b) From part (a), we know that there is an optimal solution in T.

$\forall y \in T$ J(y) = $\sum_{j\in T} h(j)d(y,j) + \sum_{j\in S} h(j)d(y,j)$

$$= \sum_{j\in(T-t)} h(j)d(y,j) + h(t)d(y,t) + \sum_{j\in S} h(j)[d(y,t)+d(t,j)]$$

$$= \sum_{j\in(T-t)} h(j)d(y,j) + h(t)d(y,t) + H(S)d(y,t) + \sum_{j\in S} h(j)d(t,j)$$

$$= \sum_{j\in(T-t)} h(j)d(y,j) + [h(t)+H(S)]d(y,t) + \sum_{j\in S} h(j)d(t,j)$$

Therefore, we have: $\forall y \in T$ J(y) = $\tilde{J}(y) + C$

where C = $\sum_{j\in S} h(j)d(t,j)$ does not depend on y

and $\tilde{J}(y) = \sum_{j\in(T-t)} h(j)d(y,j) + [h(t)+H(S)]d(y,t)$

$\tilde{J}$ is the objective function for the 1-median problem on $G'(T, A_t)$ with the weight of node t given by $H(S)+h(t)$.

(c) The isthmus edge (g, i) separates the network into two distinct subnetworks with node sets S_1 and T_1, where:

S_1 = {a, b, c, d, e, f, g, h}
T_1 = {i, j, k, l, m, n, o, p, q}

$H(S_1) = 32$ and $H(T_1) = 41$. Therefore, an optimal solution must be one of the nodes in T_1.

We now disregard the portion of the network involving nodes in S_1 and increase the weight at i to $h(i)+H(S)$ = 5 + 32 = 37. Consider the isthmus edge (i, j) which divides the new graph into two distinct subnetworks with S_2 = {j, k, l, m} and T_2 = {i, n, o, p, q}. Clearly $H(S_2) < H(T_2)$ (remember that $h(i)$ is now 37). So, we can disregard the portion of the new network involving nodes in S_2. And again, we must increase the weight at i by $H(S_2)$. So, the weight at node i becomes 37 + $H(S_2)$ = 37 + 17 = 54.

The new network consists only of nodes i, n, o, p , q, and edges between pairs of nodes from this set. Now consider the isthmus edge (i, n) which divides the new graph into nodes sets S_3 = {i} and T_3 = {n, o, p, q}. $H(S_3) = 54$ and $H(T_3) = 19$. Therefore, an optimal solution to the 1-median problem is to locate at node i.

Problem 4

(a) TD, the minimum spanning tree of D, has an even number of odd-degree nodes, like any undirected network.
Let v be the point in D that is closest to s.
We then have two cases:

- v had an even degree in TD, or
- v had on odd degree in TD.

If v had an even degree, then the addition of the edge (s, t) will make v a node of odd degree in T. This increases the number of odd-degree nodes in the "D part" of the tree T by one. We have an odd number of odd-degree nodes in R.

If v had an odd degree, then the addition of the edge (s, t) will make v a node of even degree in T. This decreases the number of odd-degree nodes in the "D part" of the tree T by 1. We then have an odd number of odd-degree nodes in R.

(b) (i) H adds one more incidence to all the odd-degree nodes in T. Therefore, the graph G has no nodes of odd-degree. It then has an Euler tour.

(ii) The key observation here is that because of the large additional cost K associated with each pairing of a point in D with a point in P, there will be only one pairing of an odd-degree point in D (call it z) with an odd-degree point in P (call it w) in the optimal matching.
(Note that from (a) we know that there will be one "left-over" odd-degree point from D and one "left-over" odd-degree point in P, after we have finished the pair wise matching of odd-degree points in D with one another and of odd-degree points in P with one another; please also note that, by construction, s will always have a degree of 2 in T).
Thus we can begin at s, find an Euler path from s to z that visits all the points in D at least once, then use the link (z, w) to go to the points in P, and then find an Euler path from w to s that visits all the points in P at least once.

Problem 5

(a) Initially (in graph R) every vertex has degree 1 (either inbound or outbound). After the pair-wise matching is added (graph G2) every vertex has degree 2. Moreover, because heads have been matched with tails, all vertices belong to (possibly disjoint cycles, i.e., all vertices can be visited as part of the Eulerian tour of each separate cycle. The "doubled tree", T, adds an even number of incidences (balanced in terms of inbound vs. outbound) to some vertices and connects all the disjoint cycles into a single graph, G4. Thus, an Eulerian tour of G4 that respects the directionality of all direct edges exists.

(b) When k=1, all vertices belong to a single cycle and an Eulerian tour that respects the directionality of the directed edges can be constructed (see part a). Moreover, the tour is of minimum length because G4 consists of the union of R (whose length always has to be traversed in its entirety, by definition) and of M, which is the most efficient way, by definition, to connect the edges in R in a legitimate way (heads visited before tails).

(c)

$$L(G2) \leq L(DTP)$$

(equality is certainly true when k=1, from part b)

$$L(T) \leq L(DTP - L(R)$$

Therefore,

$$L(G4) = L(DTP) + 2L(T) \leq 3L(DTP) - 2L(R)$$

Problem 6

(a) For L_1: according to step 2, the length between city 1 and $i_{p(1)}$ is less or equal to $\frac{1}{k}(L - 2d_{\max}) + d_{\max}$. Therefore:

$$L_1 \leq \frac{1}{k}(L - 2d_{\max}) + d_{\max} + d(i_{p(1)}, 1)$$
$$\leq \frac{1}{k}(L - 2d_{\max}) + 2d_{\max}$$

For L_j where $j \in \{2, 3, \ldots, k-1\}$, we have from step 2 that:

the length of L between 1 and $i_{p(j)}$ is less than or equal to $\frac{j}{k}(L - 2d_{\max}) + d_{\max}$,

the length of L between 1 and $i_{p(j-1)+1}$ is greater than $\frac{j-1}{k}(L - 2d_{\max}) + d_{\max}$.

Therefore, the length of L between $i_{p(j-1)+1}$ and $i_{p(j)}$ is less than or equal to $\frac{j}{k}(L - 2d_{\max})$.

This proves the results since $d(1, i_{p(j-1)+1}) + d(1, i_{p(j)}) \leq 2d_{\max}$.

For L_k, we have

$$L_k \leq L - [\frac{k-1}{k}(L - 2d_{\max}) + d_{\max}] + d_{\max} = \frac{1}{k}(L - 2d_{\max}) + 2d_{\max}$$

(b) From the previous question we have

$$L(T_{long}) \leq \frac{1}{k}(L - 2d_{\max}) + 2d_{\max} = \frac{L}{k} + 2d_{\max}\left(1 - \frac{1}{k}\right)$$

We know the following:

$L \leq \frac{3}{2}L^*$ (Christofides algorithm)

$\frac{L*}{k} \le L\left(T^*_{long}\right)$ (obvious, since $L^* \le kL\left(T^*_{long}\right)$)

$2d_{\max} \le L\left(T^*_{long}\right)$ (obvious)

Using successively the above three inequalities we have

$$\begin{aligned}
L\left(T^*_{long}\right) &\le \frac{L}{k} + 2d_{\max}\left(1-\frac{1}{k}\right) \\
&\le \frac{3L^*}{2k} + 2d_{\max}\left(1-\frac{1}{k}\right) \\
&\le \frac{3}{2}L\left(T^*_{long}\right) + 2d_{\max}\left(1-\frac{1}{k}\right) \\
&\le \frac{3}{2}L\left(T^*_{long}\right) + L\left(T^*_{long}\right)\left(1-\frac{1}{k}\right) \\
&\le \left(\frac{5}{2}-\frac{1}{k}\right)L\left(T^*_{long}\right)
\end{aligned}$$