

Structures Design Task - Engineering Analysis

1.101 Sophomore Design - Fall 2006

Each of the n shafts will be subject to bending as the top plate rotates relative to the fixed, bottom

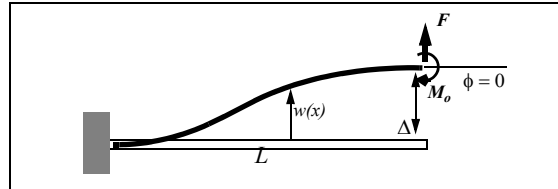
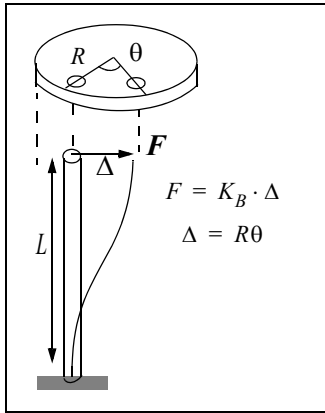


plate. For the shaft considered as a beam, we have when the slope at $x = L$ is zero.

we have

$$F = K_B \cdot \Delta \quad \text{where the stiffness} \quad K_B = \frac{12 \cdot EI}{L^3}$$

But the tip deflection, Δ , can be expressed in terms of the angle of rotation of the top plate.

$$\Delta = R\theta \quad \text{so}$$

$$F = K_B \cdot R\theta$$

The torque required to rotate the top plate about its central axis for the case when there are n identical shafts is then:

$$T = nK_B \cdot R^2\theta$$

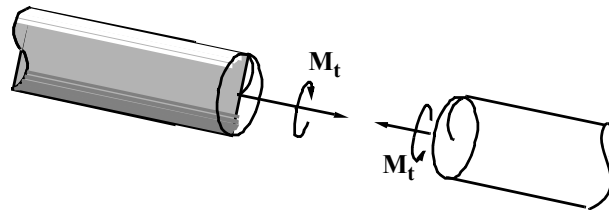
Torsion of Circular Shafts

The following has been lifted from my class notes used several years ago, in teaching 1.050. The full textbook is found under MIT's OCW site, course 1.050. The analysis of torsion of a circular shaft is organized in accord with the three main principles of the behavior of deformable medium - Compatibility of deformation, constitutive relations which bring in the material properties, and force and moment equilibrium.

Compatibility of Deformation

The cross-sections of a circular shaft in torsion rotate as if they were rigid in-plane. That is, there is no relative displacement of any two, arbitrarily chosen points of a cross section when the shaft is subjected to a torque about its longitudinal, z , axis. We prove this assertion relying on *rotational symmetry* and upon the constancy of the internal torque as we move down the axis of the shaft.

We first show that radial lines must remain straight by posing that they deform, then show a contradiction results if we do so.



Let the points along a radius take the shape of a curve in plane in the deformed state: Now consider the same set of points but from the perspective of someone who has the portion and the shaft to the right to observe. He or she would necessarily see the deformed locus as shown if the same moment is to have the same effect. Now, however, the two cannot be put back together without leaving a hole within the interior. The two displacement fields are incompatible.

The only displacement pattern that **will** fit back together is if the locus is again a straight line.

It is still possible that, while radial lines remain radial, there could be some sort of accordion effect as we march around the axis of the shaft – some radial lines coming closer together, others widening the angle between them. But no, this is not possible since we have complete rotational symmetry. Whatever happens at one angular position must happen at every other angular position.

There remains the possibility of *out of plane* bulging-out and/or dishing-in. While these would not violate rotational symmetry, we rule them out using the following argument based upon the fact that the torque does not vary as we move along the axis of the shaft.

We posit a bulging-out on a section of shaft when we apply a clock-wise moment. Running around to the other end of the section, we would claim a bulging-out there too, since the moment is again directed along the axis of the section in the same sense. But now consider the portion of the shaft to either side of the cut section. To be consistent, their cross-sections will bulge out. This is clearly an incompatible state of deformation. On the other hand, if it dished-in, then we would have a torque, the same torque producing two dramatically different effects. This argument also rules out a uniform extension or contraction of the cross-section.

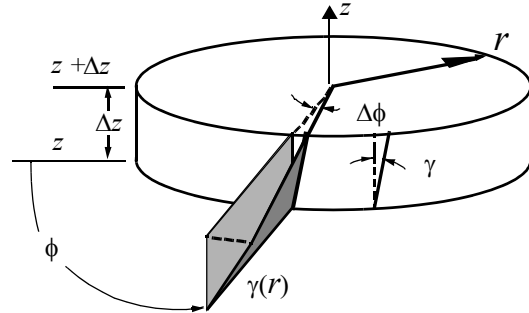
Thus, no bulging-out, no dishing-in, radial lines remain radial - a cross-section rotates as a rigid plane.

One further fact follows from the uniformity of torque at each section, namely, **the relative rotation of two cross-sections is the same for any two sections separated by the same distance along the axis of the shaft.**

If we let ϕ be the rotation of any section, then this is equivalent to saying
 $d\phi/dz$ is a constant

Consider now the strains due to the rotation of one section relative to another.

The figure shows the rotation of a section located along the axis at $z+\Delta z$ relative to a section at z just below it. Of course the section at z has rotated too, most likely. But it is the relative rotation of the two sections which gives rise to a strain, a shear strain, γ , which measures the decrease in right angle, originally formed by two line segments, one circumferential, the other axially directed as shown. From the geometry we can state:



$$\gamma = \lim_{\Delta z \rightarrow 0} (r\Delta\phi/\Delta z) = r \frac{d\phi}{dz}$$

Note that this relationship shows that the shear strain is a linear function of radius - zero at the axis, maximum at the shaft's outer radius. Note too that, with $d\phi/dz$ a constant, the shear strain does not vary with z with position along the axis of the shaft.

There are no other strains! With no deformation in plane and no bulging-out or dishing in, there are no other strains. If there existed some asymmetry like that of the truss bay structure with one diagonal member removed from each bay, then we would not be able to rule out a contraction (or extension) in the z direction.

Constitutive Relations

Because we have but one strain component, this will be a very short section. The corresponding stress is the shear stress τ and is related to the shear strain according to:

$$\tau = G\gamma = G \cdot r \left(\frac{d\phi}{dz} \right)$$

Equilibrium and the Torque-Rotation Stiffness Equation

Back in Chapter 3, we obtained an expression for resultant torque about the axis of a circular shaft due to a shear stress distribution $\tau(r)$, an arbitrary function of r . We obtained

$$M_T = 2\pi \int_0^R \tau(r)r^2 dr$$

Now, with our linear function of r , we can carry out the integration. Doing so we obtain a relationship between the applied torque M_T and the **rate of rotation** $d\phi/dz$, a *stiffness* relation.

$$M_T = 2\pi \int_0^R G \frac{d\phi}{dz} r^3 dr \quad \text{or, since } d\phi/dz \text{ and } G \text{ are constants, we are left with the integral of } r^3 \text{ and can write}$$

$$M_T = GJ \left(\frac{d\phi}{dz} \right)$$

where J is a function only of the geometry of the cross-section - its radius R . You may have encountered it as the *polar moment of inertia* $J = \int_{Area} r^2 dArea$ For the solid circular shaft $J = \pi R^4/2$.

... For a shaft of length L , the rotation of one end relative to the other is just the integral of the

constant rate of rotation over the length... We obtain then:

$$M_T = (GJ/L) \cdot \phi$$

This is our major result. Observe

- We can obtain the shear stress and strain distribution in terms of the applied moment by substitution. We obtain

$$\begin{aligned} \tau(r) &= r \cdot (M_T/J) \\ \gamma(r) &= r \cdot M_T/(GJ) \end{aligned}$$

- Our analysis is identical for a hollow shaft. All of the symmetry arguments apply. Only the expression for J changes: It becomes

$$J = \int_{Area} r^2 dA = (\pi/2)(R_o^4 - R_i^4)$$

where R_o is the outer radius and R_i the inner radius of the shaft.

- If we do anything to destroy the rotational symmetry, all bets are off. In particular if we slit a hollow tube lengthwise we dramatically decrease the torsional stiffness of the tube.

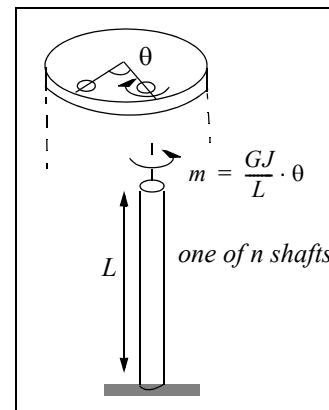
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Application to design task.

For the design task, if the top plate rotates relative to the fixed bottom plate an angle, θ , then the top end of each shaft rotates the same amount relative to the bottom fixed end. The figure shows the torque engendered in one of n shafts, *if the shaft is rigidly fixed to the top plate*. (If the shaft is free to rotate relative to the top plate, no torsional deformation, hence no moment, ensues).

For n shafts, the torque required about the central axis of the top plate is then

$$T = n K_T \theta \text{ where } K_T = GJ/L$$



For combined bending and torsion, we have, superimposing the two resistances to rotation:

$$T = n \cdot (K_B R^2 + K_T) \cdot \theta$$

where

$$K_B = \frac{12 \cdot EI}{L^3} \quad \text{and} \quad K_T = \frac{GJ}{L}$$

In these: $J = 2I = \pi r^4/2$ - for circular solid shaft. - r is the radius of the shaft, not to be confused with R .

and $G = \frac{E}{2 \cdot (1 + \nu)}$ is the “shear modulus” and ν is Poisson’s ratio which you can take as 1/3.

The above relationship between the applied torque and the stiffnesses due to bending and torsion gives you what you need to conceive of an arrangement of shafts to meet the design requirements.

Materials and stock sizes available are still to be defined, but work with the above for the moment.