1.022 Introduction to Network Models

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Lectures 22 and 23

Multiple Pure Strategy Nash Equilibria

Example: Battle of the Sexes Game.

Player 1 \ Player 2	ballet	football
ballet	**(1,4)**	$(0,0)$
football	$(0,0)$	**(4,1)**

- This game has two pure Nash equilibria.

Example: Partnership Game.

Player 1 \ Player 2	work hard	shirk
work hard	**(2,2)**	$(-1,1)$
shirk	$(1,-1)$	**(0,0)**

- Also two pure Nash equilibria.

Pareto Optimality and Social Optimality

Pareto optimality: A choice of strategies, one by each player, is Pareto-optimal if there is no other choice of strategies in which all players receive payoffs at least as high, and at least one player receives a strictly higher payoff.

Examples:

- Battle of the Sexes Game: both Nash equilibria pareto-optimal.
- Partnership Game: (work hard, work hard) pareto-optimal, (shirk,shirk) not pareto-optimal.
- Prisoner's Dilemma: Nash equilibrium not pareto-optimal. All other pair of strategies pareto-optimal.

prisoner 1 / prisoner 2	Betray	Stay silent
Betray	$(-4,-4)$	$(-1,-5)$
Stay silent	$(-5,-1)$	$(-2,-2)$

Pareto Optimality and Social Optimality

Social optimality: A choice of strategies, one by each player, is a social welfare maximizer (or socially optimal) if it maximizes the sum of the players' payoffs.

- Stronger than pareto optimality: social optimality $\Rightarrow$ pareto optimality. why?

Examples:

- Battle of the Sexes Game: both Nash equilibria socially-optimal.
- Partnership Game: (work hard, work hard) socially-optimal, (shirk,shirk) not socially-optimal.
- Prisoner's Dilemma: Nash equilibrium not socially-optimal. (Silent, Silent) socially-optimal.

Self-interested, rational behavior may or may not lead to socially optimal result.

Cournot competition

- Two firms producing a homogeneous good for the same market
- The action of a player i is a quantity, $s_i \in [0, \infty)$ (amount of good it produces).
- The utility for each player is its total revenue minus its total cost,

$$u_i(s_1, s_2) = p(s_1 + s_2) \times s_i - c \times s_i$$

  where $p(Q)$ is the price of the good (as a function of the total amount, $Q \equiv s_1 + s_2$), and c is unit cost (same for both firms).
- Assume for simplicity that $c = 1$ and $p(Q) = \max\{0, 2 - Q\}$

- A useful characterization of Nash equilibrium: an action profile s^* is a Nash equilibrium if and only if

$$s_i^* \in \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}^*).$$

- In other words, at equilibrium (s^*) each player's action (s_i^*) is a best response to the actions of other players (s_{-i}^*).
- Firm 1 faces the following optimization problem

$$\max_{s_1 \geq 0} u_1(s_1, s_2^*) = \max_{s_1 \geq 0} \; p(s_1 + s_2^*) \times s_1 - s_1,$$

where $p(s_1 + s_2^*) = \max(0, 2 - s_1 - s_2^*)$.
- A useful observation:

$$\begin{aligned} \max_{s_1 \geq 0} p(s_1 + s_2^*) \times s_1 - s_1 &= \max_{s_1 \geq 0} \; (2 - s_1 - s_2^*) \times s_1 - s_1 \quad \text{(why?)} \\ &= \max_{s_1 \geq 0} \; (1 - s_1 - s_2^*) \times s_1. \end{aligned}$$

Intersection of Best Responses

- Firm 1's decision problem:

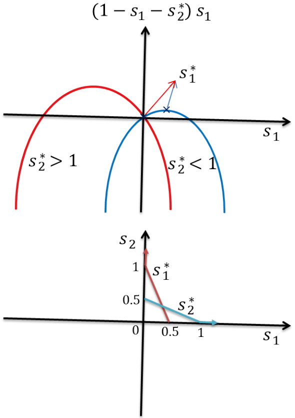
$$\max_{s_1 \geq 0} u_1(s_1, s_2^*) = \max_{s_1 \geq 0} \ (1 - s_1 - s_2^*) \times s_1.$$

- Using first order optimality condition:

$$s_1^* = \begin{cases} \frac{1-s_2^*}{2} & \text{if } s_2^* \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

- Similarly, for firm 2:

$$s_2^* = \begin{cases} \frac{1-s_1^*}{2} & \text{if } s_1^* \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$



Unique Nash equilibrium at the intersection of the two best responses: $(s_1^*, s_2^*) = (\frac{1}{3}, \frac{1}{3})$

Application: Network Cost Sharing

- A very popular strategic problem is the use of common resources.
- Use of different common resources creates congestion.
- For example, United Airlines would naturally take into account the congestion implications of using Washington Dulles as a hub.
- You might switch between different lines at supermarket checkout.
- Often in networks the problem is getting from end of the network to another.
- Whether there is a small or a large number of players, they are likely to act strategically.

Tragedy of Commons (Hardin 1968)

"The population problem has no technical solution; it requires a fundamental extension in morality." Hardin (1968).

- Herdsmen share a pasture
- If a herdsman add one more cow, he gets the whole benefit, but the cost (additional grazing) is shared by all
- Inevitably, herdsmen add too many cows, leading to overgrazing

This arises in:

- Pollution, Carbon emission
- Uncontrolled human population growth
- Overfishing
- Energy resources

Solutions:

- Privatization
- Governmental regulations
- Internalizing externalities (individual pricing)

- n individuals go out to eat
- Prior to ordering, they agree to split the check equally between all of them
- u^H: joy of eating the expensive meal, p^H: cost of the expensive meal
- u^L: joy of eating the cheap meal, p^L: cost of the cheap meal
- Whether to order the expensive or cheap dish?

Assume:

- $u^H - p^H < u^L - p^L$, $u^H - \frac{1}{n}p^H > u^L - \frac{1}{n}p^L$
- Example: $n = 3$, $u^H = 30$, $p^H = 25$, $u^L = 25$, $p^L = 15$

"Separate cheques?"

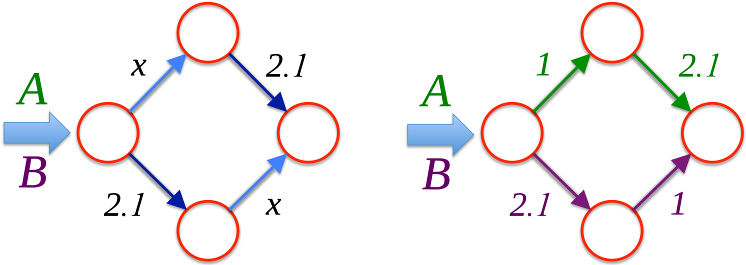
Equilibrium Analysis:

- Let x be sum of orders of others
- utility of ordering expensive: $u^H - \frac{1}{n}p^H - \frac{1}{n}x$
- utility of ordering cheap: $u^L - \frac{1}{n}p^L - \frac{1}{n}x$
- Ordering expensive is a dominant strategy
- Unique Nash equilibrium: everyone orders expensive

A Network Traffic Example

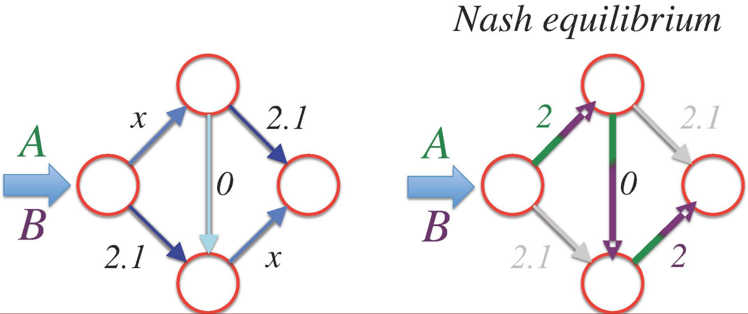
- Two players *A* and *B* each need to transfer one unit of traffic.
- Either use the upper or the lower route, minimizing their total travel time.
- Congestion times:
  - independent from the traffic flow: 2.1 for two roads
  - x where x is the flow through the road.

Nash equilibrium

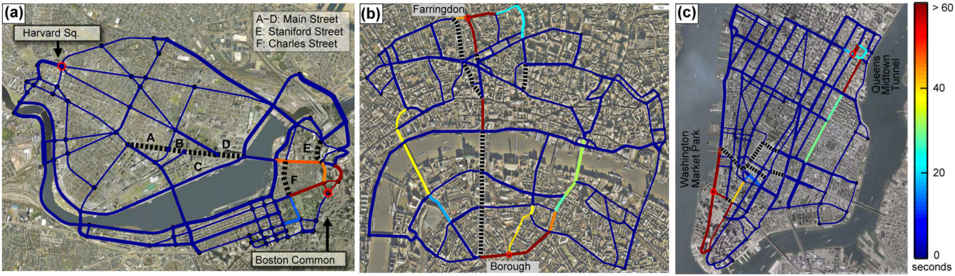


Braess' Paradox

- Idea: Addition of an intuitively helpful route.
- Paradoxical, since the addition of another route should help traffic.
- In fact, the addition of a link can never increase aggregate delay in the social optimum.
- Idea first introduced in transportation networks by Dietrich Braess in 1968.
- This is basically the prisoners' dilemma again.
- Steinberg and Zangwill '83 provided necessary and sufficient conditions for Braess paradox to occur in networks



Braess' Paradox in Cities Using Traffic Data



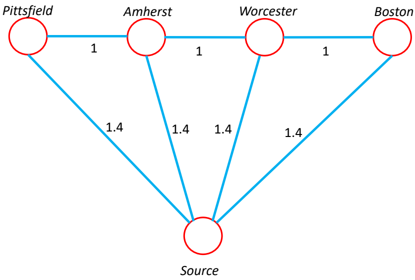
Closing streets (e.g. most of Mass. Ave in Cambridge or Blackfriars Bridge in London) marked by black dotted lines *reduces* overall congestion.[1]

[1] Youn, H. *et al.* (2009), Price of Anarchy in Transportation Networks: Efficiency and Optimality Control, *Phys. Rev. Lett.*.

- Let's define a spanning tree problem.
- There is a finite set of players $N = \{1, 2, \ldots, n\}$ (e.g. cities/municipalities) and a graph $G(V, E)$ (e.g. possible infrastructure routes) where $V = N \cup \{v_0\}$
- The players want to connect to the source node v_0
- Each edge has a cost $a : E \mapsto \mathbb{R}_{++}$, but the node gets the full benefit as long as there is a path from the node to the source node.
- The cost of each road (edge) is equally shared among the cities using the road.
- What are the equilibrium connection configurations? Are they socially optimal?

Example:

- cost of direct road: 1.4 (4 roads)
- inter city road: 1 (3 roads)

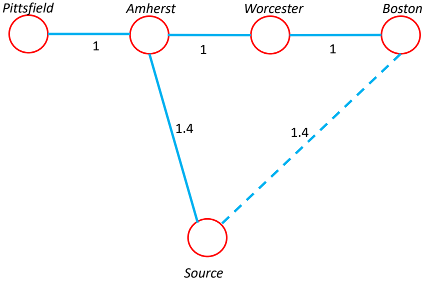


Spanning Tree Game

Example:

- cost of direct road: 1.4 (4 roads)
- inter city road: 1 (3 roads)
- Cost for Boston: $\frac{1.4}{4} + \frac{1}{2} + 1 > 1.4$
  $\Rightarrow$ Boston chooses direct link to source (not equilibrium)
- Only Boston has incentive to unilaterally deviate (why?)
- Socially optimal configuration:

$$3 + 1.4 = 4.4$$



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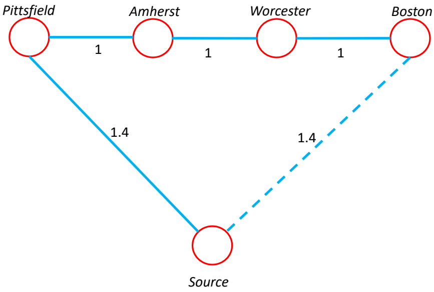
- Also socially optimal.



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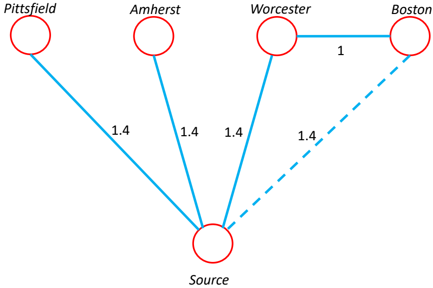
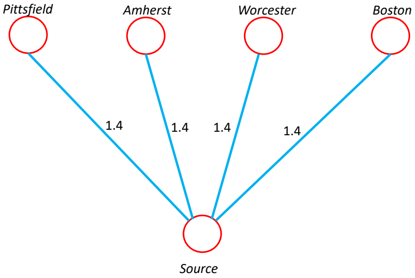


Example:

- cost of direct road: 1.4 (4 roads)
- inter city road: 1 (3 roads)
- Unique equilibrium configuration
- Not socially optimal
- Total cost: $5.6 > 4.4$



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- cost of direct road: 1.4 (4 roads)
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- Unique equilibrium configuration
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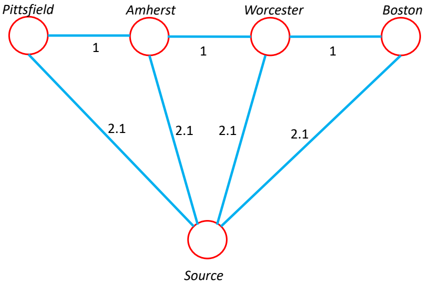
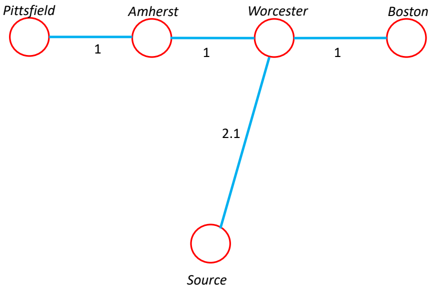
- Source (government?) imposes a tax (0.7) on direct roads.



Example:

- Source (government?) imposes a tax (0.7) on direct roads.
- Two Nash equilibria: both socially optimal
- Boston's cost: $\frac{2.1}{4} + 1 < 2.1$.
- Amherst's cost: $\frac{2.1}{4} + \frac{1}{2} < 2.1$.
- Pitsfield's cost: $\frac{2.1}{4} + \frac{1}{2} + 1 < 2.1$.



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- Pitsfield's cost: $\frac{2.1}{4} + \frac{1}{2} + 1 < 2.1$.
- Taxation benefits everyone, total cost: 5.1, source profit: 0.7
- More efficient ways to share costs (cooperative games).