5.61 Fall 2017
Problem Set #2 Solutions

1. McQuarrie, page 73, #2-6

Solution:

This question deals with solving second-order differential equations of the form $A\frac{d^2y}{dx^2}+B\frac{dy}{dx}+Cy=0$, $A, B, C \in \Re$. The standard method is to *assume* $y = e^{mx}$. If we do that, the differential equation reduces simply to $e^{mx}(Am^2 + Bm + C = 0)$. This is a quadratic equation that can be solved very easily, to give roots m_1 and m_2. The general solution to the differential equation depends on the nature of m_1 and m_2.

Case 1: $m_1 \neq m_2 \in \Re$. $y(x) = c_1e^{m_1x} + c_2e^{m_2x}$, where c_1 and c_2 are constants whose values are determined by initial conditions.

Case 2: : $m_1 = m_2 = m \in \Re$. $y(x) = e^{mx}(c_1 + c_2t)$, where c_1 and c_2 are constants whose values are determined by initial conditions.

Case 3: $m_1 = a + ib$, $m_2 = a - ib$; $a, b \in \Re$. $y(x) = e^{ax}(c_1 \sin bx + c_2 \cos bx)$, where c_1 and c_2 are constants whose values are determined by initial conditions.

Armed with these, the solutions are as follows.

a. $m_1 = -1 + i$, $m_2 = -1 - i$. Solution therefore is $e^{-x}(c_1 \sin x + c_2 \cos x)$.

b. $m_1 = 3 + 4i$, $m_2 = 3 - 4i$. Solution is therefore $e^{3x}(c_1 \sin 4x + c_2 \cos 4x)$.

c. $m_1 = -\beta + i\omega$, $m_2 = -\beta - i\omega$. Solution therefore is $e^{-\beta x}\ (c_1 \sin(\omega x) + c_2 \cos(\omega x))$.

d. $m_1 = -2 + i$, $m_2 = -2 - i$. Solution therefore is $e^{-2x}(c_1 \sin x + c_2 \cos x)$. However, we can find c_1 and c_2 as we have conditions $y(0) = c_2 = 1 \Longrightarrow c_2 = 1$. $y'(x) = e^{-2x}(-2c_1 \sin x - 2c_2 \cos x + c_1 \cos x - c_2 \sin x)$, and so $y'(0) = (-2c_2 + c_1) = -3 \Longrightarrow c_1 = -1$. Therefore, solution is $e^{-2x}(-\sin x + \cos x)$.

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2. McQuarrie, pages 76, #2-12

Solution:

We have the wave equation

$$y(x,t) = A \sin\left[\frac{2\pi}{\lambda}(x - vt)\right] \tag{1}$$

The wavelength λ is the value such that

$$y(x+\lambda,t) = y(x,t) \tag{2}$$

$$\begin{aligned}
y(x+\lambda,t) &= A\sin\left[\frac{2\pi}{\lambda}(x+\lambda-vt)\right] &(3)\\
&= A\sin\left[\frac{2\pi}{\lambda}(x-vt)+\frac{2\pi}{\lambda}\lambda\right] &(4)\\
&= A\sin\left[\frac{2\pi}{\lambda}(x-vt)+2\pi\right] &(5)\\
&= A\sin\left[\frac{2\pi}{\lambda}(x-vt)\right] = y(x,t) &(6)
\end{aligned}$$

Similarly, the frequency is the value v such that

$$y\left(x, t+\frac{1}{v}\right) = y(x,t) \tag{7}$$

$1/v$ is the period. Testing the value $v = \frac{v}{\lambda}$,

$$\begin{aligned}
y\left(x, t+\frac{\lambda}{v}\right) &= A\sin\left[\frac{2\pi}{\lambda}\left(x+\lambda-v\left[\frac{\lambda}{v}\right]\right)\right] &(8)\\
&= A\sin\left[\frac{2\pi}{\lambda}(x-vt)-\frac{2\pi}{\lambda}v\frac{\lambda}{v}\right] &(9)\\
&= A\sin\left[\frac{2\pi}{\lambda}(x-vt)-2\pi\right] &(10)\\
&= A\sin\left[\frac{2\pi}{\lambda}(x-vt)\right] = y(x,t). &(11)
\end{aligned}$$

Suppose we are looking at phase ϕ of the wave

$$\phi = \frac{2\pi}{\lambda}(x-vt) \tag{12}$$

You can understand this as looking at some point along the wave that takes on value $A\sin(\phi)$. We follow this point, so ϕ remains fixed, but we have to move along the wave (since the wave is moving). Let $\phi = 0$ for convenience Then we want to see how x changes with t at phase

$$\begin{aligned}
0 &= \frac{2\pi}{\lambda}(x-vt) &(13)\\
x &= vt &(14)\\
\frac{dx}{dt} &= v. &(15)
\end{aligned}$$

So the wave has speed v. We could have also taken any constant phase, as the derivative would have killed that constant.

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3. McQuarrie, pages 76, #2-14

Solution:

This problem demonstrates how separation of variables can help us in solving quantum mechanical problems. In particular, we will see that if we can break that Hamiltonian into a sum of known sub-Hamiltonians,

$$\widehat{\mathcal{H}} = \sum_i \widehat{\mathcal{H}}_i \tag{16}$$

where each $\widehat{\mathcal{H}}_i$ has eigenfunction ψ_i and eigenenergy E_i, then the complete solution is

$$\psi = \prod_i \psi_i \tag{17}$$

with eigenenergy

$$E = \sum_i E_i. \tag{18}$$

The differential equation of interest is the Schrödinger Equation when $V(x, y) = 0$.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \left(\frac{8\pi^2 mE}{h^2}\right) \psi(x, y) = 0 \tag{19}$$

Solving differential equations essentially boils down to guessing solutions until you are satisfied. Now, suppose we can separate ψ into

$$\psi(x, y) = \psi_x(x)\psi_y(y). \tag{20}$$

Plugging this into Equation (19) yields

$$\psi_y(y)\frac{\partial^2 \psi_x}{\partial x^2} + \psi_x(x)\frac{\partial^2 \psi_y}{\partial y^2} + \left(\frac{8\pi mE}{h^2}\right) \psi_x(x)\psi_y(y) = 0. \tag{21}$$

Divide by $\psi_x(x)\psi_y(y)$ (a mathematician is crying somewhere)

$$\frac{1}{\psi_x(x)}\frac{\partial^2 \psi_x}{\partial x^2} + \frac{1}{\psi_y(y)}\frac{\partial^2 \psi_y}{\partial y^2} + \left(\frac{8\pi mE}{h^2}\right) = 0 \tag{22}$$

The first and second terms in Equation (22) are functions of x and y respectively, and they must add to a constant. This implies that both of these functions must be equal to a constant, say K_x and K_y respectively, such that

$$K_x + K_y + \frac{8\pi mE}{h^2} = 0 \tag{23}$$

$$\frac{\partial^2 \psi_x}{\partial x^2} = K_x \psi_x \tag{24}$$

$$\frac{\partial^2 \psi_y}{\partial y^2} = K_y \psi_y \tag{25}$$

I will perform the math to solve ψ_x only, since ψ_y is the same problem, with a different label. We can in general guess solutions of the form:

$$\psi_x = c_1 e^{kx} + c_2 e^{-kx}. \tag{26}$$

Plugging this into Equation (24) yields

$$k = \sqrt{K_x}. \tag{27}$$

If $K_x > 0$, then we have an exponential solution. However, it can be shown that to fulfill the boundary conditions, the coefficients c_1 and c_2 must be zero and we have the trivial solution. Instead, suppose $K_x < 0$. Then k is complex. Write $k = iq$ where $q = \text{Im}[k]$.

$$0 = \psi(0, y) = \psi_x(0)\psi_y(0) \Longrightarrow 0 = \psi_x(0) \tag{28}$$
$$0 = \psi(a, y) = \psi_x(a)\psi_y(0) \Longrightarrow 0 = \psi_x(a) \tag{29}$$
$$0 = \psi(x, 0) = \psi_x(x)\psi_y(0) \Longrightarrow 0 = \psi_y(0) \tag{30}$$
$$0 = \psi(x, b) = \psi_x(x)\psi_y(b) \Longrightarrow 0 = \psi_y(b) \tag{31}$$

$$0 = \psi_x(0) = c_1 + c_2 \tag{32}$$

so $c_1 = -c_2$ and we can simplify

$$\psi_x = c_1(e^{iqx} - e^{-iqx}) = 2ic_1 \sin(qx) = C_1 \sin(qx) \tag{33}$$
$$0 = \psi_x(a) = C_1 \sin(qa). \tag{34}$$

This is true whenever $q = \frac{n_x \pi}{a}$ for $n_x \in \mathcal{Z}$. We can solve the y component in the same way, resulting in solutions

$$\psi_x(x) = C_1 \sin\left(\frac{n_x \pi x}{a}\right) \tag{35}$$
$$\psi_x(x) = C_3 \sin\left(\frac{n_y \pi y}{b}\right) \tag{36}$$
$$\psi(x, y) = A \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{b}\right). \tag{37}$$

A can be determined by the normalization condition. Note that $K_x = k_x^2 = (iq_x)^2$ and similarly for y. Plugging this into Equation (23)

$$\frac{8\pi^2 mE}{h^2} = -K_x - K_y \tag{38}$$
$$= \frac{n_x^2 \pi^2}{a^2} + \frac{n_y^2 \pi^2}{b^2} \tag{39}$$
$$E = \frac{h^2}{8m}\left(\frac{n_x^2}{a^2} + n_y^2 b^2\right) \tag{40}$$

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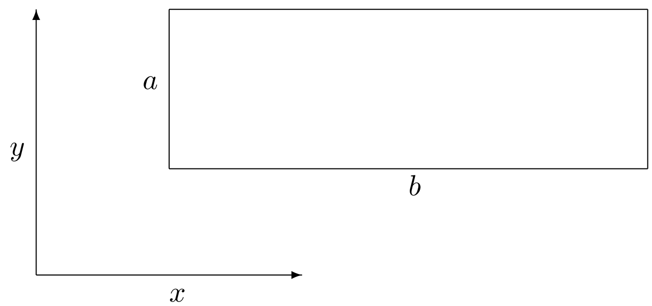
4. Consider waves on a rectangular drum membrane:



A. Show by separation of variables that the general solution to the wave equation

$$\frac{\partial^2 \mu}{\partial x^2} + \frac{\partial^2 \mu}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2},$$

has the form

$$A \sin\left(\frac{n_x \pi x}{b}\right) \sin\left(\frac{n_y \pi y}{a}\right) \cos\left(\omega_{n_x n_y} t + \phi_{n_x n_y}\right)$$

where

$$\omega_{n_x n_y} = v\pi \left[\frac{n_x^2}{b^2} + \frac{n_y^2}{a^2}\right]^{1/2}.$$

Solution: The wave equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}.$$

If we assume a factorizible solution of the form $u(x,y,t) = X(x)Y(y)T(t)$, we can rearrange terms to yield

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = \frac{1}{v^2 T(t)}\frac{\partial^2 T(t)}{\partial t^2}$$

or

$$F(x) + G(y) = V)t) \Longrightarrow F(x) + G(y) - V(t) = 0,$$

where $F(x) = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x2}$ etc. Since we have three single-variable functions (of independent variables x, y, t) summing to a constant, each of the functions must themselves be a constant. Therefore, $F(x) = -k_x^2$, $G(y) = -k_y^2$ and $V(t) = -(k_x^2 + k_y^2)$. (We choose negative numbers because the solutions corresponding to the positive numbers do not fit boundary conditions for x and y as they are exponentials, as opposed to sinusoids). Solving these equations by the method given in problem 2 (and fitting to boundary conditions) gives that $X(x) = \sin(n_x \pi \frac{x}{b})$, $Y(y) = \sin(n_y \pi \frac{y}{a})$, and $T(t) = \cos(\omega t + \phi)$ where $\omega = \pi v \left(\frac{n_x^2}{b^2} + \frac{n_y^2}{a^2}\right)$.

B.

Suggest a reason why this drum will sound awful.

Solution:

The drum will sound horrible as the frequencies will not necessarily be integer multiples of each other. For a 1D string $\omega = \frac{n\pi v}{L}$, and so ω increases in a sequence of ω_1, $2\omega_1$, $3\omega_1$, etc. However, for a square drum of side $a = b = L$, the frequencies can easily be irrational multiples of each other. For example, $\omega = \frac{\pi v}{L}(n_x^2 + n_y^2)$ means that $n_x = 1$, $n_y = 1$ leads to $\omega = \frac{\sqrt{2}\pi v}{L} = \omega_1$, but then $n_x = 2$, $n_y = 1$ leads to $\omega = \frac{\sqrt{5}\pi v}{L} = \frac{\sqrt{5}}{\sqrt{2}}\omega_1$.

This analysis can be extrapolated to any general rectangular drum, and thus we can see that there is no neat progression of frequencies, or a common time period for the modes (leading to odd interference effects) making the drum sound bad.

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5. This "magical mystery tour" problem deals with the 1-dimensional classical wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}.$$

A string of length L is anchored so that $u(x,t) = 0$ at $x = 0$ and $x = L$.

All of the answers are to be expressed in terms of L and v.

Let me start outright by saying that this is a hard problem, and there may exist alternate (and equally acceptable) solutions.

A. Write an expression for $u(x,t)$ as a linear superposition of "normal modes" of $\lambda = 2L/n$ $n = 1, 2, \ldots, \infty$.

Solution:

Any wave is a linear combination of normal modes, and so $u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi\frac{x}{L})\cos(n\pi\frac{vt}{L} + \phi_n)$

If we assume the wave was initially at rest, $u'(x,0) = 0$ which makes $\phi_n = 0$ for all n. Therefore, we can simply say: $u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi\frac{x}{L})\cos(n\pi\frac{vt}{L})$.

B. Consider the square-wave "pluck" at $t = 0$ that has the form

$$u(x,0) = 0 \qquad 0 \leq x \leq \frac{5}{8}L \text{ and } \frac{7}{8}L \leq x \leq L$$
$$u(x,0) = 1 \qquad \frac{5}{8}L < x < \frac{7}{8}L.$$

Express the pluck as an explicit linear combination of the normal modes. To do this to a good approximation you need to guesstimate the overlap integral of this square-wave pluck with each of the $n = 1 - 8$ normal modes.

Solution:

The coefficients a_n are determined by the Fourier Integral:

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L u(x,0)\sin\left(n\pi\frac{x}{L}\right)dx \\ &= \frac{2}{L}\int_{5L/8}^{7L/8} u(x,0)\sin\left(n\pi\frac{x}{L}\right)dx \text{ as } u=0 \text{ elsewhere} \\ &= \frac{2}{L}\int_{5L/8}^{7L/8} 1\times\sin\left(n\pi\frac{x}{L}\right)dx \\ &= \frac{2}{L}*\frac{L}{n\pi}\left(\cos\left(\frac{5n\pi}{8}\right)-\cos\left(\frac{7n\pi}{8}\right)\right). \end{aligned}$$

The eight term approximation, however, is nowhere close to reality (though the shape is OK):

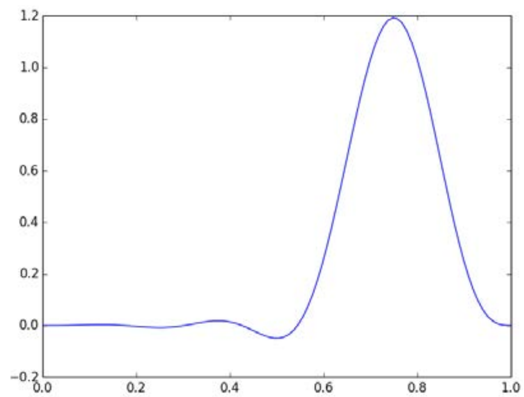
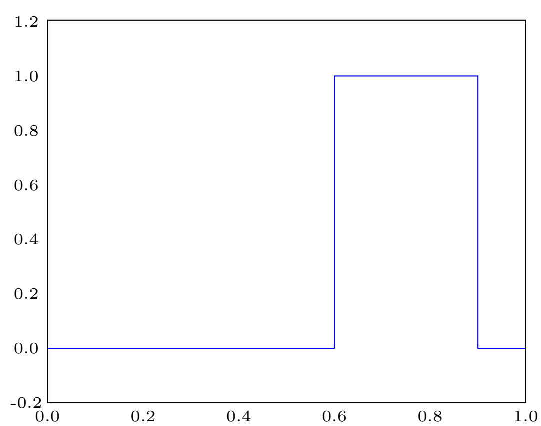


Figure 2.1: 8 term approximation.

The exact form of the pluck (1000 terms) looks as follows:



Note: Numerically doing the integral by WolframAlpha etc. is also OK.

C. Identify the 3 normal modes that make the largest contributions to this $u(x, 0)$ pluck and write the 3-term sum approximation of the moving wave, $u(x, t)$.

Solution:

The three largest coefficients are a_1, a_2, a_3. $\frac{1}{n}$ means that coefficients a_n decrease as n increases, and so the first few will be the largest (note, $a_4 = 0$, and is out of the running-a_5 makes a bigger contribution). Please note that it is the absolute value of the coefficient that matters, not the sign (since there is a cosine factor that evolves with time, and changes signs). The numbers are $a_1 = 0.345$; $a_2 = -0.450$; $a_3 = 0.227$ and $a_5 = -0.166$.

D. What is the earliest time, $t_{\text{recurrence}}$, when $u(t_{\text{recurrence}}) \approx u(x, 0)$? *Sketch* the half-recurrence wave, $u(x, t_{\text{recurrence}}/2)$.

Solution: The n$^{\text{th}}$ mode has angular frequency $\omega_n = \frac{n\pi v}{L}$. Therefore, they have a time period of $T_n = \frac{2\pi}{\omega_n} = \frac{2L}{nv}$. From this we can see that all waves will have undergone an integer number of oscillations in time $t = \frac{2L}{v}$, as $t = nT_n$ (t is an integer multiple of all recurrence times). Therefore, recurrence time is $\frac{2L}{v}$–Time period of the First Harmonic.

At $t = \frac{L}{v}$, the eight term approximation of the wave gives us:

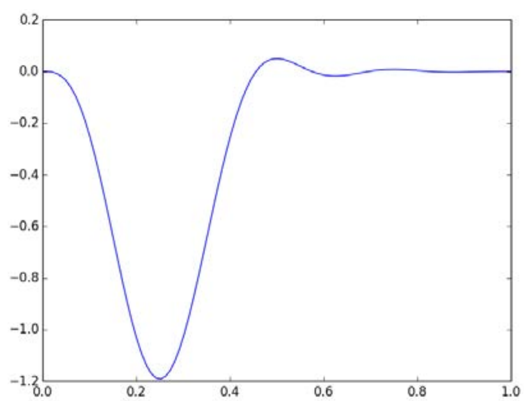
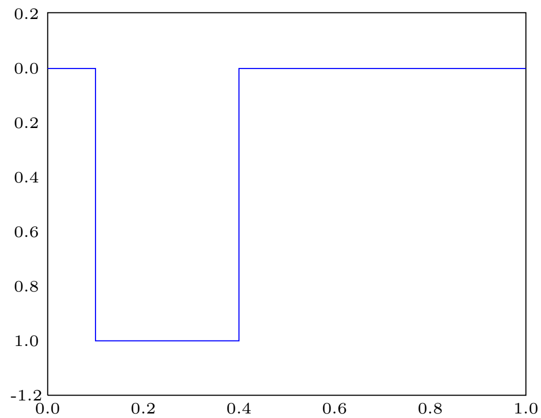


Figure 2.3: 8 term approximation at half recurrence time.

The exact is:



The overall effect is that the wave is reflected along both the x and y axes, taking the midpoint of the string to be the origin.

E. *(optional)* Make an eleven frame time-lapse movie of $u(x,t)$ for $t = m\left(\frac{t_{\text{recurrence}}}{10}\right)$ $m = 0, 1, \ldots, 10$. It is OK (preferable) to hand-sketch rather than plot an explicit mathematical expression. The important thing is that all of the qualitative features should be present in your sketch.

F. *(optional)* By comparing some features of the $m = 0$ and 1 frames of the movie, estimate the velocity of the traveling wave.

Solution:

Not discussed in details as optional. The trick is to pretend that the standing poioipwave is a sum of two traveling waves moving in opposite directions, and then let those traveling waves move apart (and get rejected from the ends). Please contact Professor Field if you want more details.

G. Using the approximate superposition from part **C**, compute the time-dependent quantity $\langle x \rangle_t = \int_0^L xu(x,t)dx$ and plot $\langle x \rangle_t$ and $\frac{d}{dt}\langle x \rangle_t$ for the time interval $0 \leq t \leq t_{\text{recurrence}}$. It is OK to guesstimate these quantities, but explain your reasoning.

Solution:

The plots are as follows (with 8 term approximation, the exact looks similar):

$\langle x \rangle$:

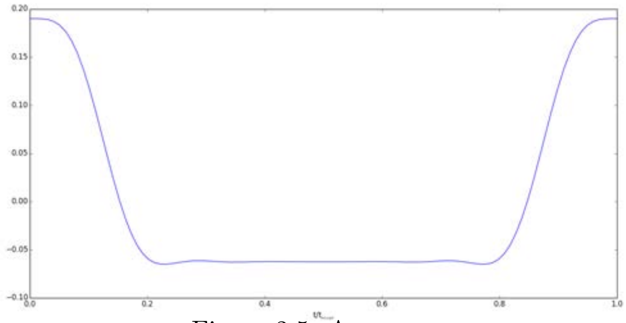


Figure 2.5: Average x

$\frac{d\langle x \rangle}{dt}$:

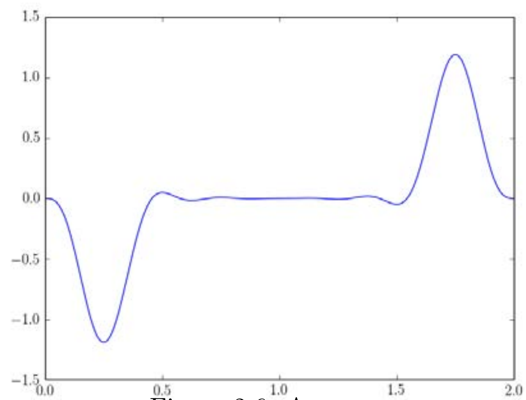


Figure 2.6: Average v

The three–term plots also look remarkably similar, and all qualitative features are preserved.

H. What do the plots in part **G** tell you about the evolution of the specific pluck? (Words like dephasing, rephasing, velocity, and spreading will be very welcome in your answer to this question.)

Solution:

Dephasing: If one observes the $\langle x \rangle$ plot, it is immediately noticed that the wave moves away from its original average x value to something more random (where average x is almost 0 due to different phases at different parts). This shows that the highly regular wave shape is rapidly lost.

Rephasing: After one full period however, the wave returns to its original shape.

Velocity: The centre of the wavepacket is not stationary-it moves as the $\frac{d\langle x \rangle}{dt}$ plot shows.

Spreading: The wave spreads and covers larger area, as dephasing happens and shape is lost. This is harder to visualize without the snapshots at different moments, but it suffices to say that the wave is not static.

I. *(optional)* Suppose a *spatially narrower* pluck

$$u(x,0) = 1 \qquad \frac{11}{16}L < x < \frac{13}{16}L,$$

or a *centered* pluck,

$$u(x,0) = 1 \qquad \frac{3}{8}L < x < \frac{5}{8}L,$$

were chosen. Do not actually derive an expression for this pluck! Suggest reasons for the qualitative differences between the time evolution of these two plucks and that of the pluck documented in parts **B** through **H**?

Solution:

Symmetric pluck: There would not be any even harmonics present (wrong symmetry makes Fourier integral zero). Thus, the wave will actually go flat at times.

Narrower pluck: Less interference and dephasing. More regular wave movement.

6.

A. Find the energies (E_n) and normalized wavefunctions (ψ_n) for a particle in an infinite (symmetric) box

$U(x) = 0 \qquad -L/2 < x < L/2$

$U(x) = \infty \qquad |x| \geq L/2$.

Solution:

$$E_n = \frac{n^2h^2}{8mL^2}. \ \psi_n(x) = \sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L} \text{ if } n \text{ is odd and } \psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \text{ if } n \text{ is even.}$$

B. Relate the E_n and ψ_n for problem **6.A** to those for the (zero-left-edge) box.

$U(x) = 0 \qquad 0 < x < L$

$U(x) = \infty \qquad x \leq 0, x \geq L$.

Define a simple coordinate transformation (e.g., $x' = ax + b$) that makes the $\{\psi_n\}$ for **6.A** look like those of **6.B**.

Solution:

$$E_n = \frac{n^2h^2}{8mL^2}.\ \psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \text{ for all } n.$$

Energy is unaffected by translation, and $x' = x + \frac{L}{2}$. If you take the wavefunction from part **A**, and insert x' instead of x there, you will recover the wavefunction for part **B**.

C. What happens to E_n and ψ_n if the box of **6.A** is raised to higher energy

$U(x) = E_0 > 0 \qquad |x| < L/2$

$U(x) = \infty \qquad |x| \geq L$?

This *should not require* a repeat of a complete calculation analogous to that in **6.A**.

Solution:

$$E_n = E_0 + \frac{n^2h^2}{8mL^2}.\ \psi_n(x) = \sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L} \text{ if } n \text{ is odd and } \psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \text{ if } n \text{ is even.}$$

ψ is unchanged if we raise the bottom of the box by a constant amount (since we can fix the potential baseline arbitrarily), but the energy levels are moved up by a constant E_0.

D. Write a transformation (eg. $x' = ax + b$) that enables you to obtain the $\{\psi_n\}$ for

$U(x) = 0 \qquad |x| < L/2$

$U(x) = \infty \qquad |x| \geq L$

from the $\{\psi_n\}$ of **6.A**. However, this box is twice as long as the box in **6.A**.

Solution:

$$E_n = \frac{n^2h^2}{32mL^2}.\ \psi_n(x) = \sqrt{\frac{1}{L}}\cos\frac{n\pi x}{2L} \text{ if } n \text{ is odd and } \psi_n(x) = \sqrt{\frac{1}{L}}\sin\frac{n\pi x}{2L} \text{ if } n \text{ is even.}$$

Transformation $x' = \frac{x}{2}$. To be completely fair, this is not correct, as doubling the length changes the normalization constant too. The question is a bit vague and ill defined overall.

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7. For the particle in the zero-left-edge box of 6.B:

A. Compute the probability of finding the particle in the interval

$$\frac{0.999}{2}L \leq x \leq \frac{1.001}{2}L$$

for $n = 1, 2, 3$, and 10^4.

Solution:

If $P(a < x < b)$ is the probability of finding a particle between $x = a$ and $x = b$, then the Born interpretation of the wavefunction says that:

$$P = \int_a^b \psi^*(x)\psi(x),$$

where $\psi(x)$ is the normalized wavefunction for the particle, and $\psi^*(x)$ is the complex conjugate of $\psi(x)$. Since $\psi(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$ for a particle in a box

$$\begin{aligned} P(a < x < b) = P &= \frac{2}{L} \int_a^b \sin^2 \frac{n\pi x}{L} dx \\ &= \frac{1}{L} \int_a^b \left(1 - \cos \frac{2n\pi x}{L}\right) dx \\ &= \frac{b-a}{L} + \frac{1}{2n\pi} \left(\sin \frac{2n\pi a}{L} - \sin \frac{2n\pi b}{L}\right). \end{aligned}$$

After putting in $a = \frac{0.999L}{2}$ and $b = \frac{1.001L}{2}$, we get:

1. $n = 1 \implies P = 0.002$
2. $n = 2 \implies P = 6 \times 10^{-9}$. 0 is OK too.
3. $n = 3 \implies P = 0.002$
4. $n = 10^4 \implies P = 0.001$ This suggests that the particle is uniformly distributed, which corresponds to the near-classical behavior expected because n is so large.

B. Compute $\langle x \rangle$ and $\langle p \rangle$ for $n = 1, 2, 3$, and 10^4.

[To a *very good approximation* this should not require evaluation of any integrals.]

Solution:

Symmetry forces $\langle x \rangle = \frac{L}{2}$ for all n. $\langle p \rangle = 0$ for a particle in a box as this is a stationary state, and there is no reason to favor one direction over the other.

C. Compute $\Delta x \Delta p$ for $n = 1, 2, 10^4$, where Δx is the "uncertainty" in x. It is the square root of the variance $\Delta x = [\langle x^2 \rangle - \langle x \rangle^2]^{1/2}$ and $\Delta p = [\langle p^2 \rangle - \langle p \rangle^2]^{1/2}$ **Hint**: the values of $\langle x \rangle$, $\langle p \rangle$, and $\langle p^2 \rangle$

do not require evaluation of any integrals. Evaluation of $\langle x^2 \rangle$ *will* require use of integral tables or some other cleverness.

Solution:

For $\left\langle p^2 \right\rangle$:

$$\frac{n^2h^2}{8mL^2} = E = \langle T \rangle \text{ as } V = 0, \text{ energy is purely kinetic, and so } E = \text{Average KE}$$
$$= \frac{p^2}{2m}$$
$$\Longrightarrow \left\langle p^2 \right\rangle = \frac{n^2h^2}{4L^2} \text{ and } \Delta p = \frac{nh}{2L}.$$

$\left\langle x^2 \right\rangle = \frac{L^2}{3} - \frac{L^2}{2n^2\pi^2}$ by doing the integral $\frac{2}{L}\int_a^b x^2 \sin^2 \frac{x\pi x}{L} dx$ by parts. Therefore $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{L}{2n\pi}\sqrt{\frac{n^2\pi^2}{3} - 2}$.

Thus $\Delta x \Delta p = \frac{\hbar}{2}\sqrt{\frac{n^2\pi^2}{3} - 2}$.

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8. *(optional)* Consider a 2-slit experiment with the following characteristics:

slits: 1 cm high, 0.01 cm wide

slit separation: 0.2 cm

distance to screen: 100 cm

wavelength of light: 500nm

area of screen: 10 cm × 10 cm

Discuss (there is no simple correct answer) how to specify a light intensity in Watts that ensures only one photon at a time is "interacting" with the screen. How long does it take for one photon to travel from slit to screen?

Solution:

A good starting point would be to ask how long would it take for one photon to travel from slit to screen.
