**5.61 Fall 2017**
**Problem Set #2**

## 1. McQuarrie, page 73, #2-6

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## 2. McQuarrie, pages 76, #2-12

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## 3. McQuarrie, pages 76, #2-14

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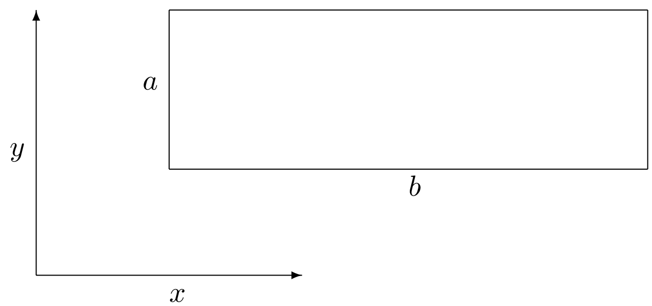
## 4. Consider waves on a rectangular drum membrane:




**A.** Show by separation of variables that the general solution to the wave equation

$$\frac{\partial^2 \mu}{\partial x^2} + \frac{\partial^2 \mu}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2},$$

has the form

$$A \sin\left(\frac{n_x \pi x}{b}\right) \sin\left(\frac{n_y \pi y}{a}\right) \cos\left(\omega_{n_x n_y} t + \phi_{n_x n_y}\right)$$

where

$$\omega_{n_x n_y} = v\pi \left[\frac{n_x^2}{b^2} + \frac{n_y^2}{a^2}\right]^{1/2}.$$

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**B.**

Suggest a reason why this drum will sound awful.

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## 5. This "magical mystery tour" problem deals with the 1-dimensional classical wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}.$$

A string of length $L$ is anchored so that $u(x, t) = 0$ at $x = 0$ and $x = L$.

*All of the answers are to be expressed in terms of L and v.*

Let me start outright by saying that this is a hard problem, and there may exist alternate (and equally acceptable) solutions.

**A.** Write an expression for $u(x, t)$ as a linear superposition of "normal modes" of $\lambda = 2L/n$ $n = 1, 2, \ldots, \infty$.

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**B.** Consider the square-wave "pluck" at $t = 0$ that has the form

$$u(x, 0) = 0 \qquad 0 \leq x \leq \frac{5}{8}L \text{ and } \frac{7}{8}L \leq x \leq L$$
$$u(x, 0) = 1 \qquad \frac{5}{8}L < x < \frac{7}{8}L.$$

Express the pluck as an explicit linear combination of the normal modes. To do this to a good approximation you need to guesstimate the overlap integral of this square-wave pluck with each of the $n = 1 - 8$ normal modes.

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**C.** Identify the 3 normal modes that make the largest contributions to this $u(x, 0)$ pluck and write the 3-term sum approximation of the moving wave, $u(x, t)$.

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**D.** What is the earliest time, $t_{\text{recurrence}}$, when $u(t_{\text{recurrence}}) \approx u(x, 0)$? *Sketch* the half-recurrence wave, $u(x, t_{\text{recurrence}}/2)$.

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**E.** *(optional)* Make an eleven frame time-lapse movie of $u(x, t)$ for $t = m\left(\frac{t_{\text{recurrence}}}{10}\right)$ $m = 0, 1, \ldots, 10$. It is OK (preferable) to hand-sketch rather than plot an explicit mathematical expression. The important thing is that all of the qualitative features should be present in your sketch.

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**F.** *(optional)* By comparing some features of the $m = 0$ and 1 frames of the movie, estimate the velocity of the traveling wave.

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**G.** Using the approximate superposition from part **C**, compute the time-dependent quantity $\langle x \rangle_t = \int_0^L xu(x,t)dx$ and plot $\langle x \rangle_t$ and $\frac{d}{dt}\langle x \rangle_t$ for the time interval $0 \leq t \leq t_{\text{recurrence}}$. It is OK to guesstimate these quantities, but explain your reasoning.

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**H.** What do the plots in part **G** tell you about the evolution of the specific pluck? (Words like dephasing, rephasing, velocity, and spreading will be very welcome in your answer to this question.)

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**I.** *(optional)* Suppose a *spatially narrower* pluck

$$u(x,0) = 1 \qquad \frac{11}{16}L < x < \frac{13}{16}L,$$

or a *centered* pluck,

$$u(x,0) = 1 \qquad \frac{3}{8}L < x < \frac{5}{8}L,$$

were chosen. Do not actually derive an expression for this pluck! Suggest reasons for the qualitative differences between the time evolution of these two plucks and that of the pluck documented in parts **B** through **H**?

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## 6.

**A.** Find the energies ($E_n$) and normalized wavefunctions ($\psi_n$) for a particle in an infinite (symmetric) box

$U(x) = 0 \qquad -L/2 < x < L/2$

$U(x) = \infty \qquad |x| \geq L/2$ .

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**B.** Relate the $E_n$ and $\psi_n$ for problem **6.A** to those for the (zero-left-edge) box.

$U(x) = 0 \qquad 0 < x < L$

$U(x) = \infty \qquad x \leq 0, x \geq L$ .

Define a simple coordinate transformation (e.g., $x' = ax + b$) that makes the $\{\psi_n\}$ for **6.A** look like those of **6.B**.

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**C.** What happens to $E_n$ and $\psi_n$ if the box of **6.A** is raised to higher energy

$U(x) = E_0 > 0 \qquad |x| < L/2$

$U(x) = \infty \qquad |x| \geq L$ ?

This *should not require* a repeat of a complete calculation analogous to that in **6.A**.

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**D.** Write a transformation (eg. $x' = ax + b$) that enables you to obtain the $\{\psi_n\}$ for

$U(x) = 0 \qquad |x| < L/2$

$U(x) = \infty \qquad |x| \geq L$

from the $\{\psi_n\}$ of **6.A**. However, this box is twice as long as the box in **6.A**.

## 7. For the particle in the zero-left-edge box of 6.B:

**A.** Compute the probability of finding the particle in the interval

$$\frac{0.999}{2}L \leq x \leq \frac{1.001}{2}L$$

for $n = 1, 2, 3$, and $10^4$.

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**B.** Compute $\langle x \rangle$ and $\langle p \rangle$ for $n = 1, 2, 3$, and $10^4$.

[To a *very good approximation* this should not require evaluation of any integrals.]

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**C.** Compute $\Delta x \Delta p$ for $n = 1, 2$, $10^4$, where $\Delta x$ is the "uncertainty" in $x$. It is the square root of the variance $\Delta x = [\langle x^2 \rangle - \langle x \rangle^2]^{1/2}$ and $\Delta p = [\langle p^2 \rangle - \langle p \rangle^2]^{1/2}$ **Hint**: the values of $\langle x \rangle$, $\langle p \rangle$, and $\langle p^2 \rangle$ do not require evaluation of any integrals. Evaluation of $\langle x^2 \rangle$ *will* require use of integral tables or some other cleverness.

## 8. *(optional)* Consider a 2-slit experiment with the following characteristics:

slits: 1 cm high, 0.01 cm wide

slit separation: 0.2 cm

distance to screen: 100 cm

wavelength of light: 500nm

area of screen: 10 cm $\times$ 10 cm

Discuss (there is no simple correct answer) how to specify a light intensity in Watts that ensures only one photon at a time is "interacting" with the screen. How long does it take for one photon to travel from slit to screen?