1.1.3 Matrix addition and matrix/vector multiplication

For the linear system of N equations for N unknowns:

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = b_2 \\ \vdots \\ \vdots \\ a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N = b_N \end{array} \qquad \textbf{(1.1.3-1)}$$

expressed in matrix/vector for as

$$A\,\underline{x} = \underline{b} \qquad \textbf{(1.1.3-2)}$$

We know that from $\oint$1.1.2 how to manipulate the N-dimensional real vectors $\underline{x}, \underline{b}, \in \mathbf{R}^N$.

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ : \\ : \\ x_n \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ : \\ : \\ b_n \end{bmatrix} \qquad \textbf{(1.1.3-3)}$$

We write the matrix A as

$$\begin{array}{cl} A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ : & : & : & : & & : \\ : & : & : & : & & : \\ a_{n1} & a_{n2} & \ldots & a_{nj} & \ldots & a_{nn} \end{bmatrix} & \begin{array}{l} \\ \longleftarrow \\ 2^{nd}\text{ row} \\ \\ \\ \end{array} \\ \begin{array}{c} \uparrow \\ j\text{th column} \end{array} & \end{array}$$

a_{ij} = element of A in row #I and column #j.

If the number of columns (N) equals the number of rows (N), A is called a <u>square matrix.</u>

To describe the size of a matrix with M rows and N columns, it is common to call it a M "by" N or M x N matrix.

How do we manipulate matrices? First, look at some simple operations.

-multiplication of a M x N matrix A by a scalar c:

$$cA = c\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ : & : & : & \\ : & : & : & \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} & \dots & ca_{1N} \\ ca_{21} & ca_{22} & \dots & ca_{2N} \\ : & : & : & \\ : & : & : & \\ ca_{M1} & ca_{M2} & \dots & ca_{MN} \end{bmatrix} \qquad \textbf{(1.1.3-5)}$$

-Addition of a M x N matrix A with a M x N matrix B:

$$A + B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ : & : & : & \\ : & : & : & \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1N} \\ b_{21} & b_{22} & \dots & b_{2N} \\ : & : & : & \\ : & : & : & \\ b_{M1} & b_{M2} & \dots & b_{MN} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1N}+b_{1N} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2N}+b_{2N} \\ : & : & : & \\ : & : & : & \\ a_{M1}+b_{M1} & a_{M2}+b_{M2} & \dots & a_{MN}+b_{MN} \end{bmatrix} \qquad \textbf{(1.1.3-6)}$$

Note that A + B = B + A **(1.1.3-7)** and that two matrices can be added <u>only</u> if both the number of rows and columns of each matrix are the same.

Other properties, such as c(A + B) = cA + cB are easily established.

-Multiplication of a N x N matrix A with an N-dimensional vector $\underline{v}$:
If we are to have equivalence of notation between the set of linear algebraic equations **(1.1.3-1)** and the matrix/vector equation **(1.1.3-2)**, written explicitly below as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ : & : & : & & : \\ : & : & : & & : \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2] \\ : \\ : \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ : \\ : \\ b_n \end{bmatrix} \qquad \textbf{(1.1.3-9)}$$

then the rule for multiplying (to be accurate, pre-multiplying) an N-dimensional vector $\underline{v}$ by an N x N matrix A must be:

$$A\underline{v} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ : & : & : & & : \\ : & : & : & & : \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2] \\ : \\ : \\ v_N \end{bmatrix} = \begin{bmatrix} a_{11}v_{11} & a_{12}v_2 & \ldots & a_{1N}v_N \\ a_{21}v_1 & a_{22}v_2 & \ldots & a_{2N}v_N \\ : & : & : & \\ : & : & : & \\ a_{N1}v_1 & a_{N2}v_2 & \ldots & a_{NN}v_N \end{bmatrix} \qquad \textbf{(1.1.3-10)}$$

We see that $A\underline{v}$ is also an N-dimensional vector, whose jth component (i.e. the value in the jth row of $A\underline{v}$) is

$$(A\underline{v}) = a_{j1}v_1 + a_{j2}v_2 + \ldots + a_{jN}v_N = \sum_{k=1}^{N} a_{jk}v_k \qquad \textbf{(1.1.3-11)}$$

This formula defines a summation of products across row #j of the matrix and down the vector,

$$\begin{bmatrix} \\ \\ \rightarrow a_{jk} \rightarrow \\ \\ \\ \end{bmatrix} \begin{bmatrix} \\ \downarrow \\ v_k \\ \downarrow \\ \\ \end{bmatrix} \Rightarrow (A\underline{v})_j$$

-Multiplication of an M x N matrix A with an N-dimensional vector $\underline{v}$:

From the rule for $A\underline{v}$ just presented, it is clear that the number of columns of A must equal the number of elements of $\underline{v}$, but we can also define $A\underline{v}$ when $M \neq N$.

$$A\underline{v} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2N} \\ : & : & : & & : \\ : & : & : & & : \\ a_{M1} & a_{M2} & a_{M3} & \ldots & a_{MN} \end{bmatrix}}_{\text{M x N matrix A}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ : \\ : \\ v_N \end{bmatrix}}_{\text{N-dimensional Vector } \underline{v}} = \underbrace{\begin{bmatrix} a_{11}v_{11} & a_{12}v_2 & \cdots & a_{1N}v_N \\ a_{21}v_1 & a_{22}v_2 & \cdots & a_{2N}v_N \\ : & : & : & \\ : & : & : & \\ a_{M1}v_1 & a_{M2}v_2 & \cdots & a_{MN}v_N \end{bmatrix}}_{\text{M-dimensional vector}}$$

(1.1.3-12)

For example,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 11 & 12 & 13 & 14 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 30 \\ 20 \\ 130 \end{bmatrix} \qquad \textbf{(1.1.3-13)}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 4 & 5 & 6 \\ 5 & 6 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 11 \\ 32 \\ 29 \end{bmatrix} \qquad \textbf{(1.1.3-14)}$$

Note also that:

$$A(c\underline{v}) = cA\underline{v} \qquad\qquad A(\underline{v}+\underline{w}) = A\underline{v} + A\underline{w} \qquad \textbf{(1.1.3-15)}$$