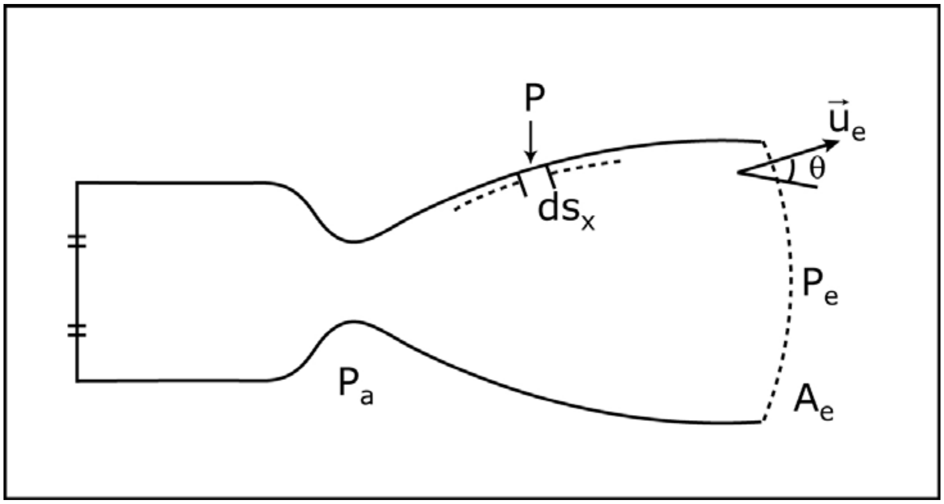
Lecture 2: Rocket Nozzles and Thrust

Rocket Thrust (Thermal rockets)



$$\dot{m} = \iint_{A_e} \rho\, u_n dA_e$$

$$\iint_{\substack{Solid\,int.\\ surfaces}} P\ dS_x - \iint_{A_e} P_e dA_{e_x} = \iint_{A_e} u_x \underbrace{\left(\rho\, u_n\right) dA_e}_{d\dot{m}}$$

(Tanks included)

Note: $\iint_{s.,int} P_a dS_x - \iint_{Ae} P_a dA_{e_x} = 0$, so subtract,

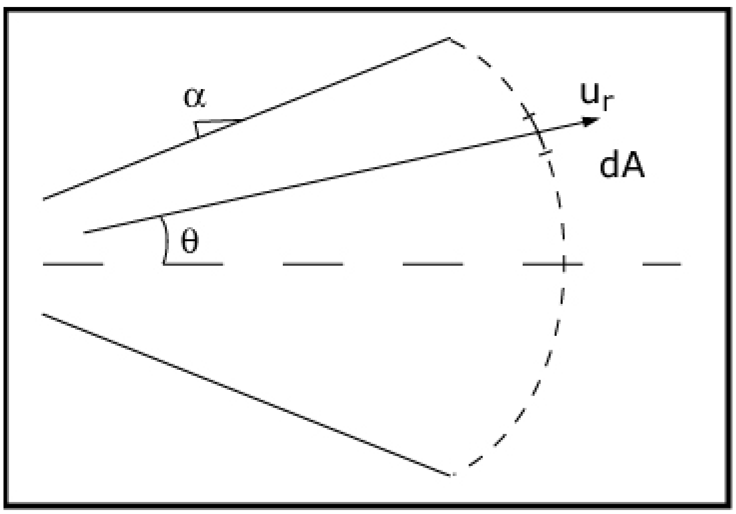
$$\underbrace{\iint_{Solid\,int.} \left(P - P_a\right)\ dS_x}_{Thrust\ \equiv F} = \iint_{A_e} \left(P_e - P_a\right) dA_{e_x} + \iint_{A_e} \rho u_x u_n dA_e$$

In general then, define $\overline{u}_e = \dfrac{\iint_{A_e} \rho u_x u_n dA_e}{\iint_{A_e} \rho u_n \, dA_e}$

and $\overline{P}_e = \dfrac{\iint_{A_e} P_e dA_{e_x}}{A_{e_x}}$

$$\Rightarrow \boxed{F = \dot{m}\,\overline{u}_e + \left(\overline{P}_e - P_a\right) A_{e_x}}$$

If things are nearly constant on <u>spherical</u> caps, modify control volume to spherical wedge:



$$\dot{m} = \iint_{A_e} \rho u_r dA$$

$$\iint_{\substack{\text{int.}\\ solids.}} \left(P - P_a\right) dS_x - \iint_{A_e} \left(P_e - P_a\right) dA_{e_x} = \iint_{A_e} \left(\rho u_r\right) u_x dA$$

$dA_{e_x} = dA \cos\theta \qquad u_x = u_r \cos\theta$

Define

$$\bar{u}_e = \frac{\iint_{A_e} \rho u_r u_x dA}{\dot{m}} \quad ; \quad \bar{P}_e = \frac{\iint_{A_e} P_e dA_{e_x}}{A_{e_x}}$$

and use

$$dA = 2\pi r \sin\theta \; r d\theta$$

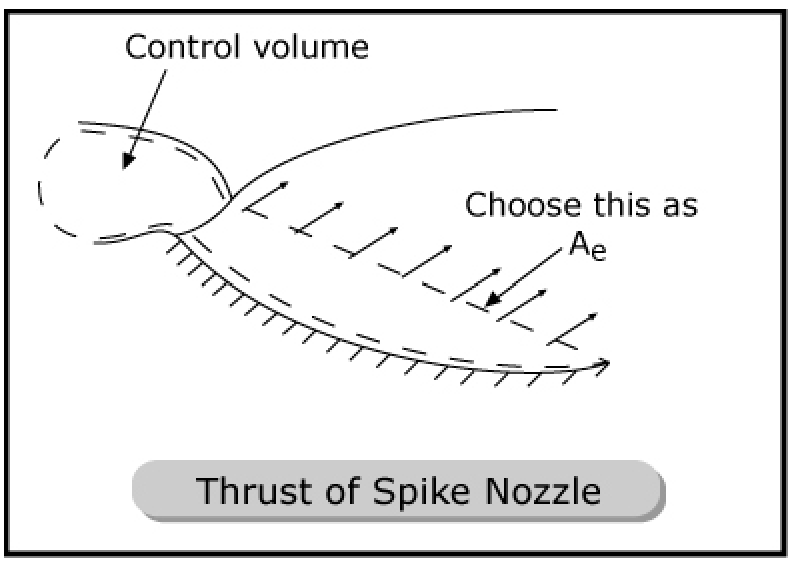
For ideal conical flow, ρ, u_r, P are constant over A_e. Then

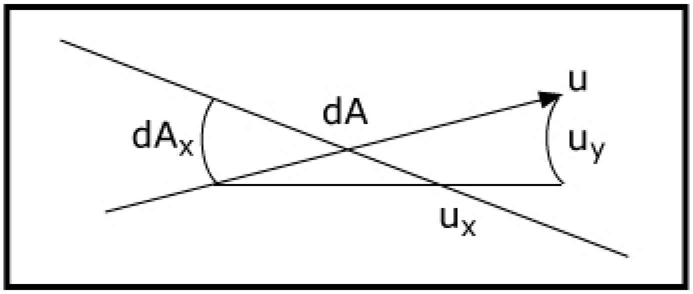
$$\bar{u}_e = \frac{\rho u_r^2 \iint_{A_e} \cos\theta dA}{\rho u_r \iint_{A_e} dA} = u_r \frac{\int_0^\alpha 2\pi r \sin\theta \cos\theta d\theta}{\int_0^\alpha 2\pi r \sin\theta d\theta} = u_r \frac{\frac{1}{2}\sin^2\alpha}{1-\cos\alpha}$$

or

$$\boxed{u_e = u_r \frac{1+\cos\alpha}{2}}$$

Also, since $P_e = const$ on the exit surface, $\boxed{\bar{P}_e = P_e}$





$$\underbrace{\int_{s.s.} \left(P - P_a\right) dA_x}_{F} + \int_{A_e} \left(P_e - P_a\right) dA_x = \int_{A_e} \left(\rho u_n dA\right) u_x$$

$$F = \dot{m}\,\bar{u}_e + \left(\bar{P}_e - P_a\right) A_{e_x}$$

$$\dot{m} = \int_{A_e} \rho u_n dA$$

$$\bar{u}_e = \frac{\int_{A_e} \rho u_n u_x dA}{\int_{A_e} \rho u_n \, dA}$$

$$\bar{P}_e = \frac{\int_{A_e} P_e dA_x}{\int_{A_e} dA_x}$$

$$A_x = \int_{A_e} dA_x$$

At design, $P_e = P_a$ (and parallel flow beyond). Also u_{e_x}

Then uniform $\rightarrow F = \dot{m}\, u_{e_x}$

Energy Considerations

So, momentum balance gives the Thrust Equation. What does an Energy Balance give?

Start with a near-stagnant flow in the upstream plenum ("combustion chamber", or "nuclear heater" or "arc heated plenum"). The total specific enthalpy $h_{tc} = h_c + \frac{1}{2}\upsilon_c^2 \cong h_c$ may be different for different streamlines, due to combustion "streaks:, arc constriction, etc., But along the flow expansion in the nozzle, h_t is conserved for each streamline. At the exit,

$$h_e + \frac{1}{2}\upsilon_e^2 = h_{t_o} \qquad \text{(each streamtube)}$$

or

$$\upsilon_e = \sqrt{2\left(h_{t_c} - h_e\right)} \cong \sqrt{2\left(h_c - h_e\right)}$$

For a well-expanded nozzle, with large area ratio, he$\rightarrow$o by adiabatic expansion, and υ_e tend to a max. $\upsilon_{e\,MAX} = \sqrt{2\,h_{t_c}}$. In any real, finite expansion, he$\neq$o, so some of h_{t_c} is wasted as thermal energy in the exhaust. Define a nozzle efficiency.

$$\eta_N = \frac{h_{t_c} - h_e}{h_{t_c}} = 1 - \frac{h_e}{h_{t_c}} \cong 1 - \frac{h_e}{h_c}$$

For ideal gas, $\frac{h_e}{h_c} = \frac{T_e}{T_c} = \left(\frac{P_e}{P_c}\right)^{\frac{\gamma-1}{\gamma}}$. But, in any case,

$$\upsilon_e = \upsilon_{e\,MAX}\sqrt{\eta_N} = \sqrt{\eta_N}\sqrt{2\,h_{t_c}} \qquad \left(i.e.,\ \eta_N = \frac{\upsilon_e^2/2}{h_{t_c}}\right)$$

Since $P_e \cong$ uniform, so is η_N , even when h_{t_c} is not. Also, υ_e is non-uniform if h_{t_c} is (in proportion to $\sqrt{h_{t_c}}$).

The Jet Power is the kinetic energy flow out of the nozzle

$$P_{Jet} = \frac{1}{2}\dot{m}\left(h_{t_c} - h_e\right) = \eta_N\, h_{t_c}\, \dot{m}$$

Effect of Stagnation Enthalpy Non-uniformities

Consider a case where h_{t_c} varies from streamtube ($d\dot{m}$) to streamtube (but P_e=const., so η_N = const.). Then

$$F = \iint \upsilon_e \, d\dot{m} + \left(P_e - P_a\right) A_e$$

For P_a = o (vacuum operation) and P_aA_e << F (large expansion), (or if $P_e = P_a$)

$$F \cong \iint \upsilon_e \, d\dot{m} = \sqrt{2\,\eta_N} \iint \sqrt{h_{t_c}} \, d\dot{m} \qquad (1)$$

and the input power is $P = \iint h_{t_c} \, d\dot{m}$ (2)

<u>It can be shown that</u> $\left\{\begin{matrix} \text{P is minimum (For a given F, } \dot{m}) \\ \text{or F is maximum} \left(\text{given P, } \dot{m}\right) \end{matrix}\right\}$ if the flow is <u>uniform</u> (h_{t_c} =const.). If it were, we would have

$$F_{UNIF.} = \sqrt{2\,\eta_N} \; \dot{m} \, \sqrt{h_{t_c}} \quad ; \quad P_{UNIF} = \dot{m} \, h_{t_c}$$

Eliminating h_{t_c}, $P_{UNIF} = \dot{m} \left(\dfrac{F_{UNIF}}{\sqrt{2\,\eta_N \; \dot{m}}} \right)^2 = \dfrac{F_{UNIF}^2}{2\,\eta_N \; \dot{m}} = \dfrac{F^2}{2\,\eta_N \; \dot{m}}$

Define an "efficiency" $\eta_{UNIF} = \dfrac{P_{UNIF}}{P_{ACTUAL}}$ (for a given thrust)

Now, express in general F by (1) and P by (2)

$$\eta_{UNIF} = \frac{\left(\sqrt{2\,\eta_N} \iint \sqrt{h_{t_c}} d\dot{m}\right)^2}{2\,\eta_N \left(\iint d\dot{m}\right)\left(\iint h_{t_c} \, d\dot{m}\right)}$$

Define "generalized vectors" $\underset{\sim}{u} = 1$ $\quad \underset{\sim}{\upsilon} = \sqrt{h_{t_c}}$ in the space of the dm values.

Then $\eta_{UNIF} = \dfrac{\left(\underset{\sim}{u} \cdot \underset{\sim}{\upsilon}\right)^2}{\left|\underset{\sim}{u}^2\right| \cdot \left|\underset{\sim}{\upsilon}^2\right|} \le 1 \quad \left(= \cos^2 \theta_{\underset{\sim}{u} \cdot \underset{\sim}{\upsilon}}\right)$.

Equality applies only when $\underset{\sim}{\upsilon}$ is a constant, i.e., h_{t_c} =const. This proves the "ansatz".

<u>Example</u>:

- 50% of flow has $h_{t_c} = 0.5\ \bar{h}_{t_c}$
- 50% of flow has $h_{t_c} = 1.5\ \bar{h}_{t_c}$

$$\eta_{UNIF} = \frac{\left(\frac{1}{2}\sqrt{0.5} + \frac{1}{2}\sqrt{1.5}\right)^2}{(1)\left(\frac{1}{2}0.5 + \frac{1}{2}1.5\right)} = 0.933$$ (6.7% energy loss due to nonunif.

Important in arcjets, less in film-cooled chemical rockets.