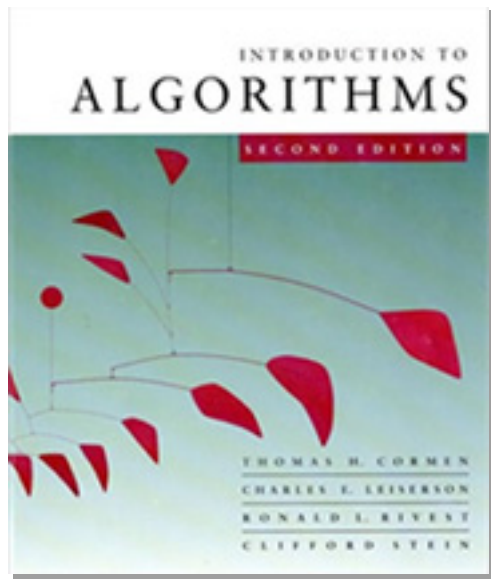


# *Introduction to Algorithms*

6.046J/18.401J



## LECTURE 2

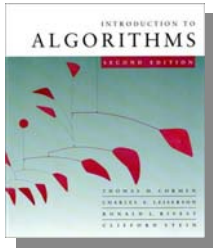
### Asymptotic Notation

- $O$ -,  $\Omega$ -, and  $\Theta$ -notation

### Recurrences

- Substitution method
- Iterating the recurrence
- Recursion tree
- Master method

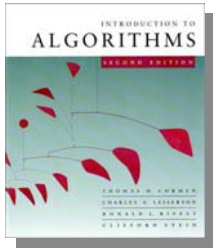
**Prof. Erik Demaine**



# Asymptotic notation

$O$ -notation (upper bounds):

We write  $f(n) = O(g(n))$  if there exist constants  $c > 0$ ,  $n_0 > 0$  such that  $0 \leq f(n) \leq cg(n)$  for all  $n \geq n_0$ .

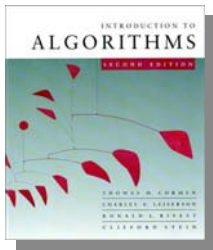


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**EXAMPLE:**  $2n^2 = O(n^3)$  ( $c = 1$ ,  $n_0 = 2$ )



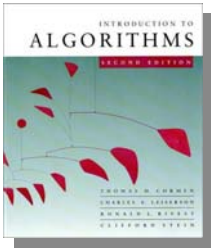
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*functions,  
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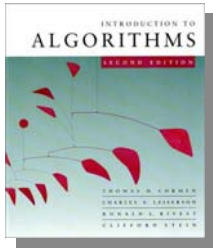
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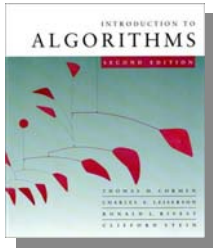
*functions,  
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*funny, “one-way”  
equality*



# Set definition of $O$ -notation

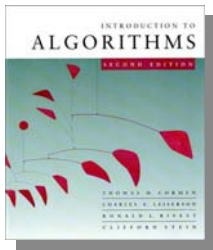
$O(g(n)) = \{ f(n) : \text{there exist constants } c > 0, n_0 > 0 \text{ such that } 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0 \}$



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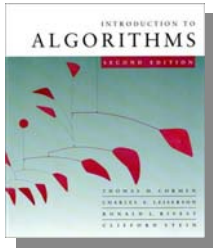
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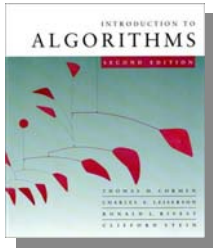
(*Logicians:*  $\lambda n. 2n^2 \in O(\lambda n. n^3)$ , but it's convenient to be sloppy, as long as we understand what's *really* going on.)





# Macro substitution

***Convention:*** A set in a formula represents an anonymous function in the set.



# Macro substitution

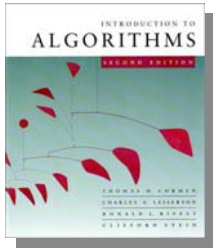
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**EXAMPLE:**  $f(n) = n^3 + O(n^2)$

means

$$f(n) = n^3 + h(n)$$

for some  $h(n) \in O(n^2)$ .



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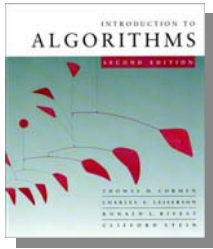
***EXAMPLE:***  $n^2 + O(n) = O(n^2)$

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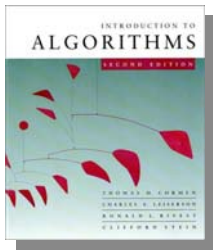
$$n^2 + f(n) = h(n)$$

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# $\Omega$ -notation (lower bounds)

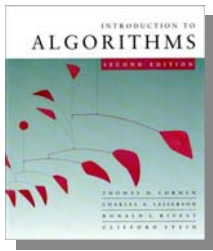
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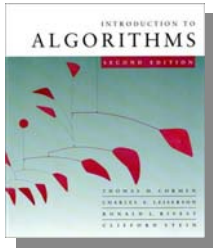


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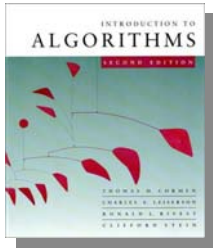
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**EXAMPLE:**  $\sqrt{n} = \Omega(\lg n)$  ( $c = 1, n_0 = 16$ )



# $\Theta$ -notation (tight bounds)

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$$

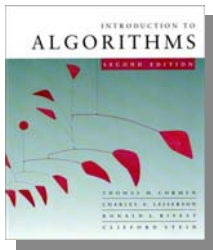


# $\Theta$ -notation (tight bounds)

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$$

**EXAMPLE:**  $\frac{1}{2}n^2 - 2n = \Theta(n^2)$



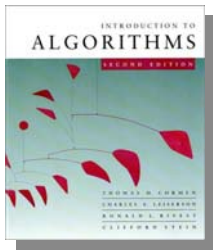


# $o$ -notation and $\omega$ -notation

$O$ -notation and  $\Omega$ -notation are like  $\leq$  and  $\geq$ .  
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$o(g(n)) = \{ f(n) : \text{for any constant } c > 0, \text{ there is a constant } n_0 > 0 \text{ such that } 0 \leq f(n) < cg(n) \text{ for all } n \geq n_0 \}$

**EXAMPLE:**  $2n^2 = o(n^3)$  ( $n_0 = 2/c$ )

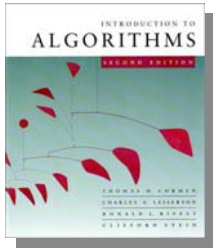


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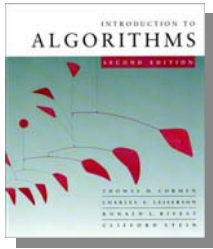
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**EXAMPLE:**  $\sqrt{n} = \omega(\lg n)$  ( $n_0 = 1 + 1/c$ )



# Solving recurrences

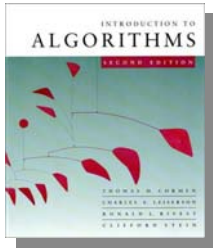
- The analysis of merge sort from *Lecture 1* required us to solve a recurrence.
- Recurrences are like solving integrals, differential equations, etc.
  - Learn a few tricks.
- *Lecture 3*: Applications of recurrences to divide-and-conquer algorithms.



# Substitution method

*The most general method:*

- 1. *Guess*** the form of the solution.
- 2. *Verify*** by induction.
- 3. *Solve*** for constants.



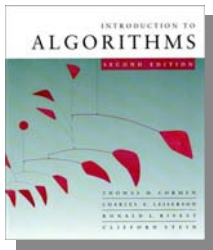
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**EXAMPLE:**  $T(n) = 4T(n/2) + n$

- [Assume that  $T(1) = \Theta(1)$ .]
- Guess  $O(n^3)$ . (Prove  $O$  and  $\Omega$  separately.)
- Assume that  $T(k) \leq ck^3$  for  $k < n$ .
- Prove  $T(n) \leq cn^3$  by induction.

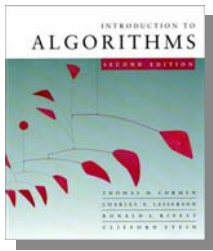


# Example of substitution

$$\begin{aligned}T(n) &= 4T(n/2) + n \\ &\leq 4c(n/2)^3 + n \\ &= (c/2)n^3 + n \\ &= cn^3 - ((c/2)n^3 - n) \leftarrow \textit{desired} - \textit{residual} \\ &\leq cn^3 \leftarrow \textit{desired}\end{aligned}$$

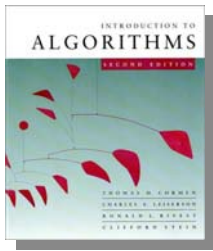
whenever  $(c/2)n^3 - n \geq 0$ , for example,  
if  $c \geq 2$  and  $n \geq 1$ .

*residual*



# Example (continued)

- We must also handle the initial conditions, that is, ground the induction with base cases.
- **Base:**  $T(n) = \Theta(1)$  for all  $n < n_0$ , where  $n_0$  is a suitable constant.
- For  $1 \leq n < n_0$ , we have “ $\Theta(1)$ ”  $\leq cn^3$ , if we pick  $c$  big enough.



# Example (continued)

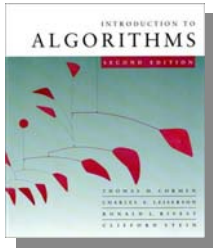
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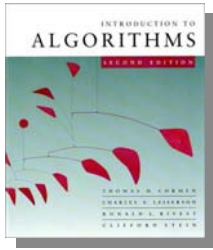
***This bound is not tight!***





# A tighter upper bound?

We shall prove that  $T(n) = O(n^2)$ .

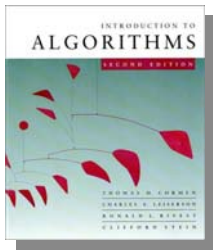


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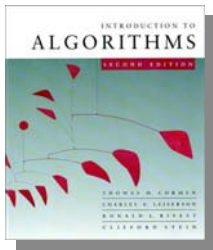
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~~$O(n^2)$~~  **Wrong!** We must prove the I.H.



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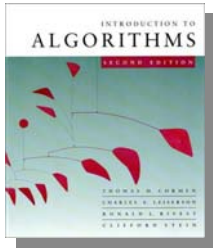
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~~$= O(n^2)$~~  **Wrong!** We must prove the I.H.

$$= cn^2 - (-n) \quad [ \text{desired} - \text{residual} ]$$

$\leq cn^2$  for **no** choice of  $c > 0$ . Lose!

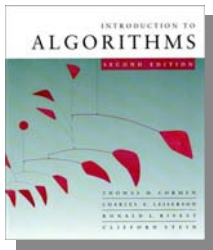


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**IDEA:** Strengthen the inductive hypothesis.

- *Subtract* a low-order term.

*Inductive hypothesis:*  $T(k) \leq c_1k^2 - c_2k$  for  $k < n$ .



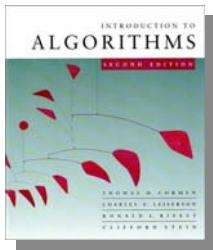
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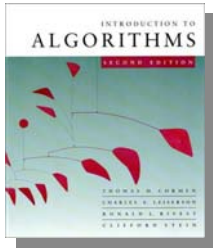
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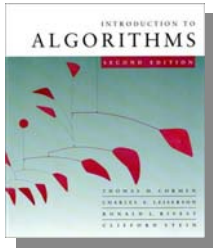
Pick  $c_1$  big enough to handle the initial conditions.



# Recursion-tree method

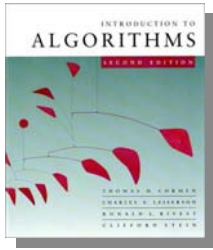
- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (...).
- The recursion-tree method promotes intuition, however.
- The recursion tree method is good for generating guesses for the substitution method.





# Example of recursion tree

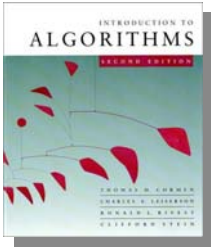
Solve  $T(n) = T(n/4) + T(n/2) + n^2$ :



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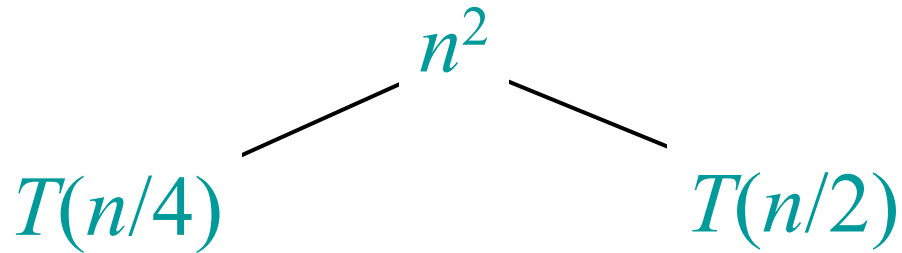
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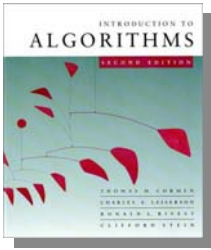
$$T(n)$$



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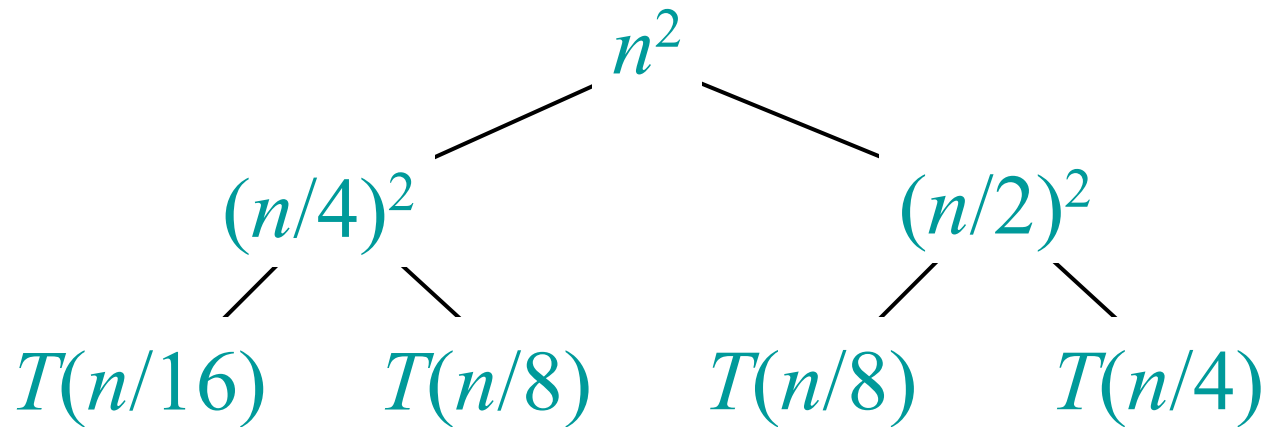
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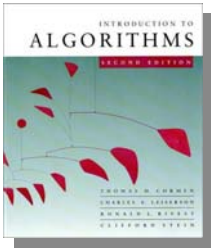




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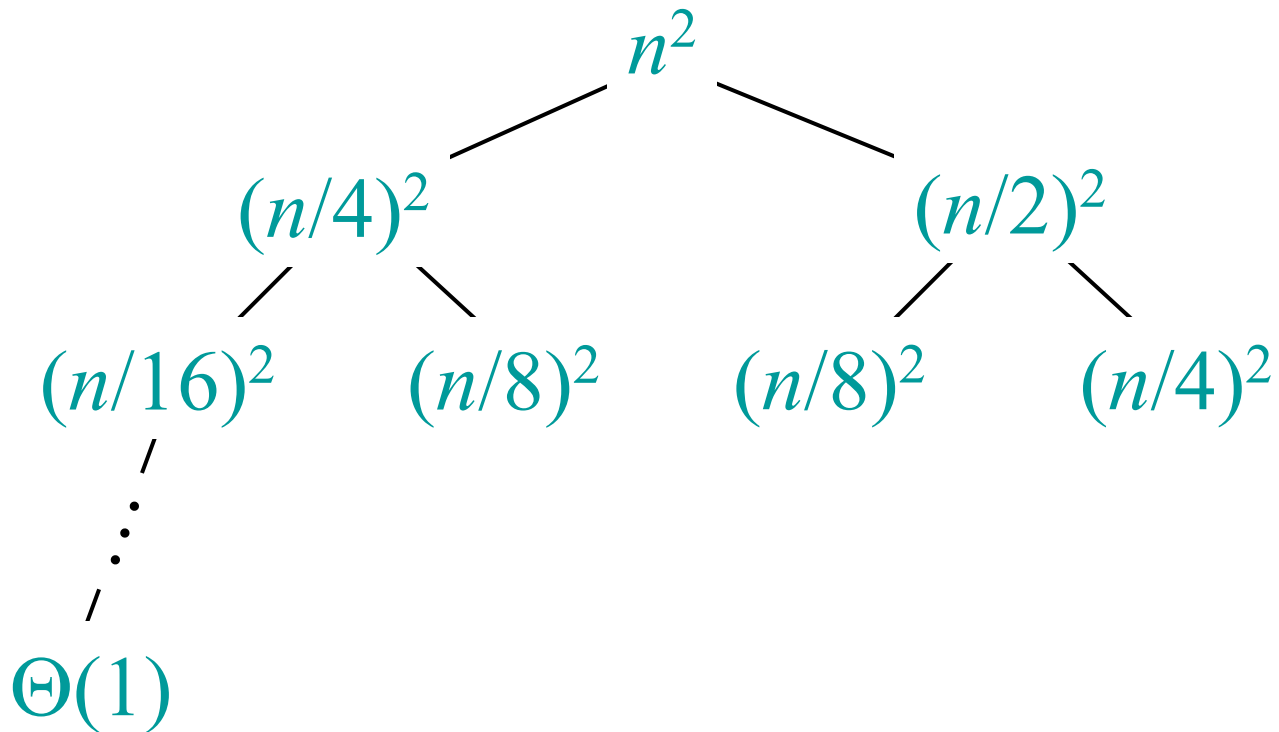
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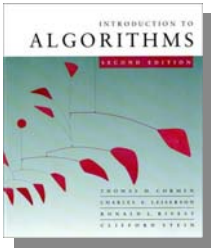




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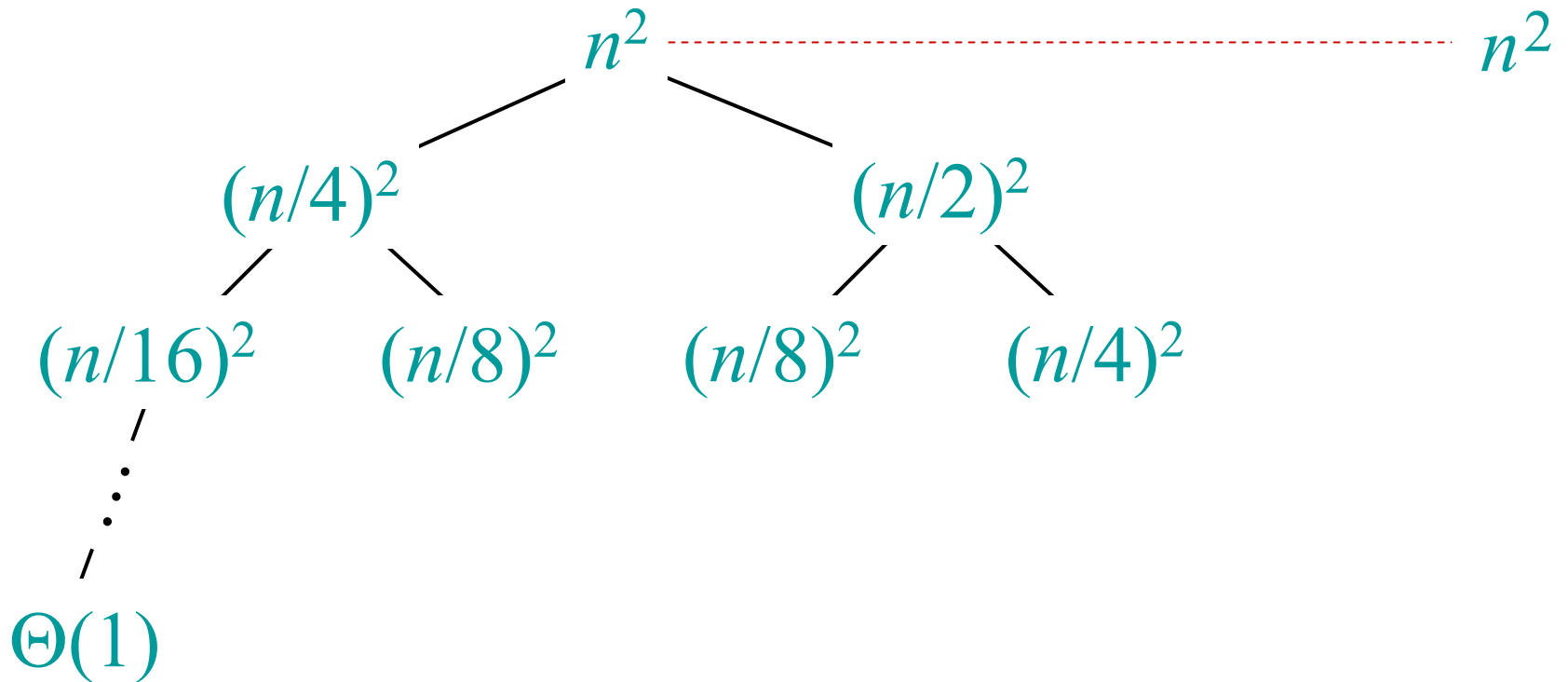
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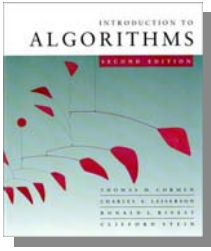




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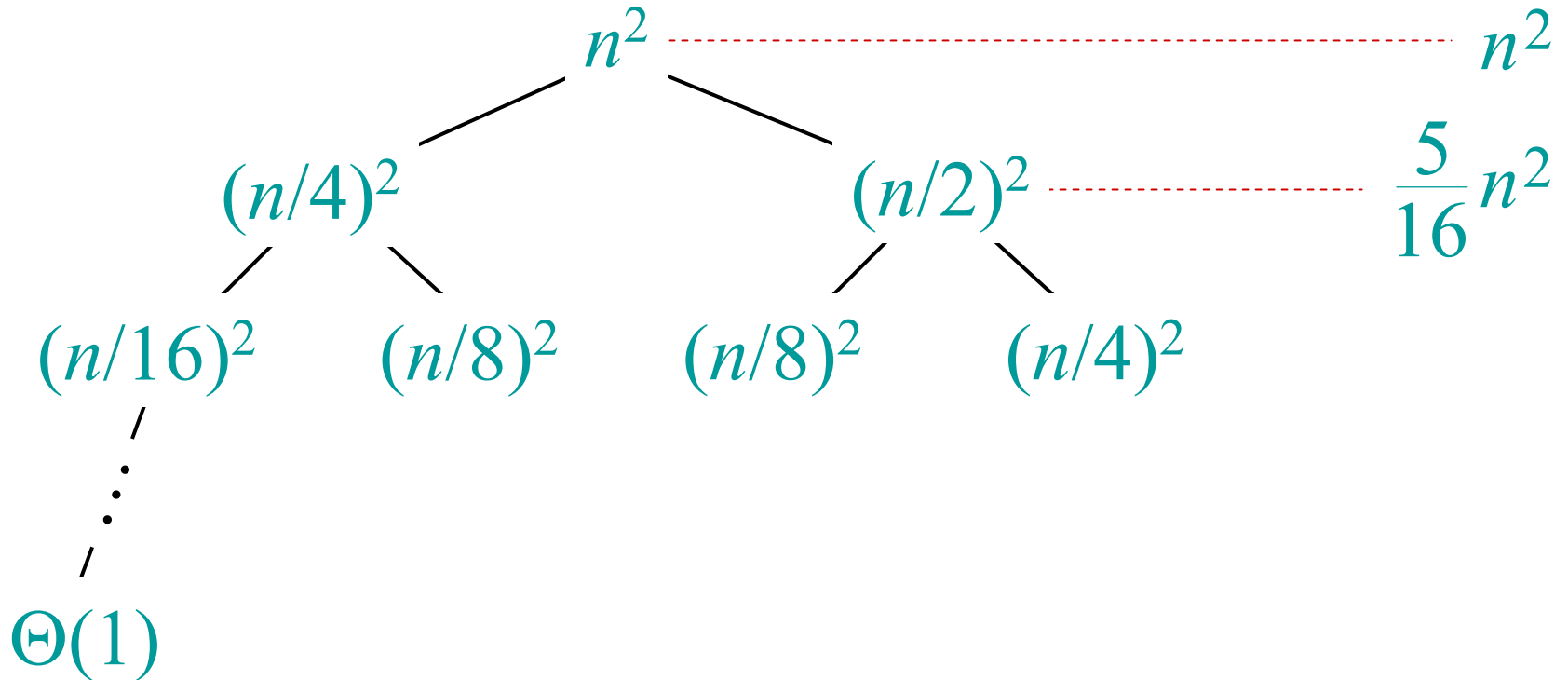
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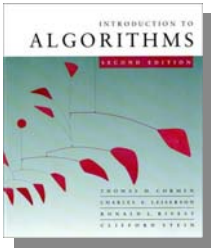




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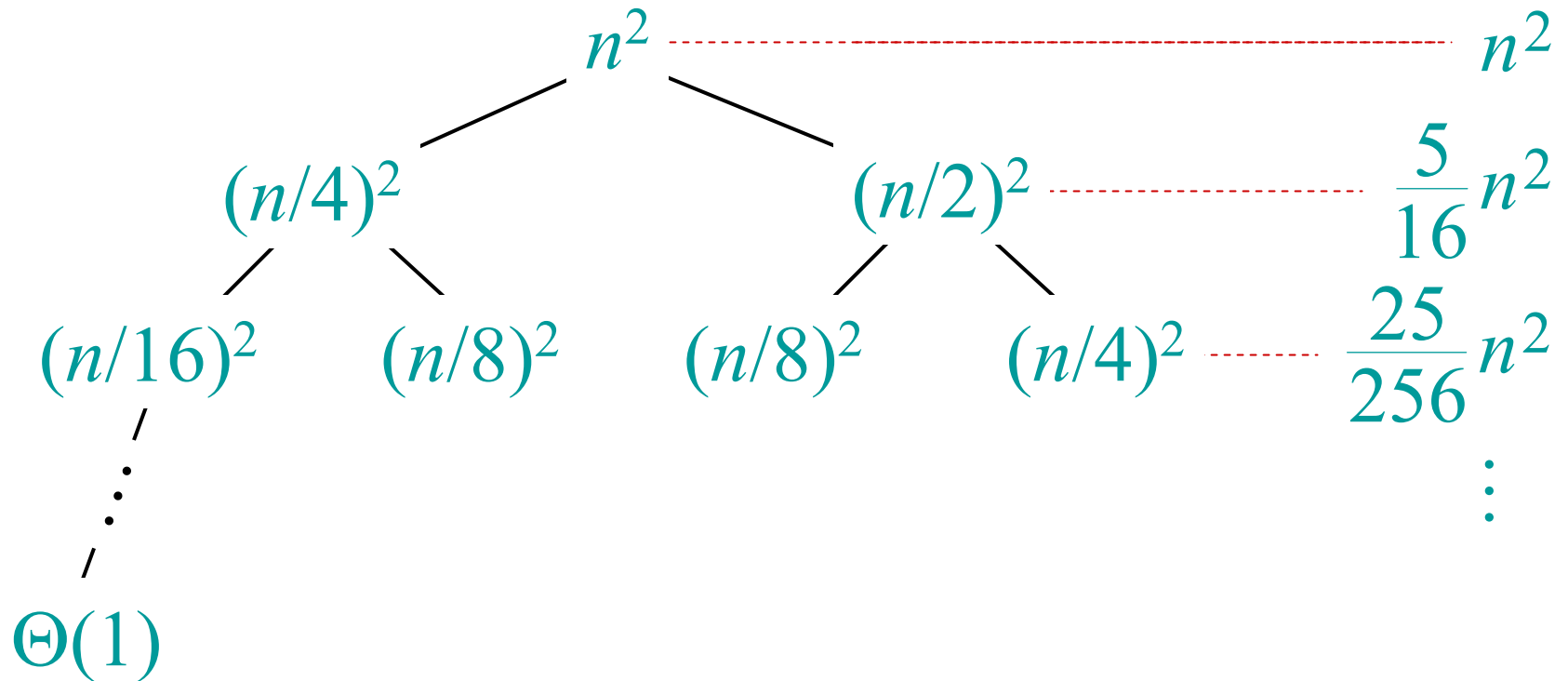
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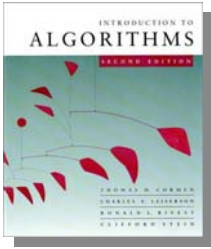


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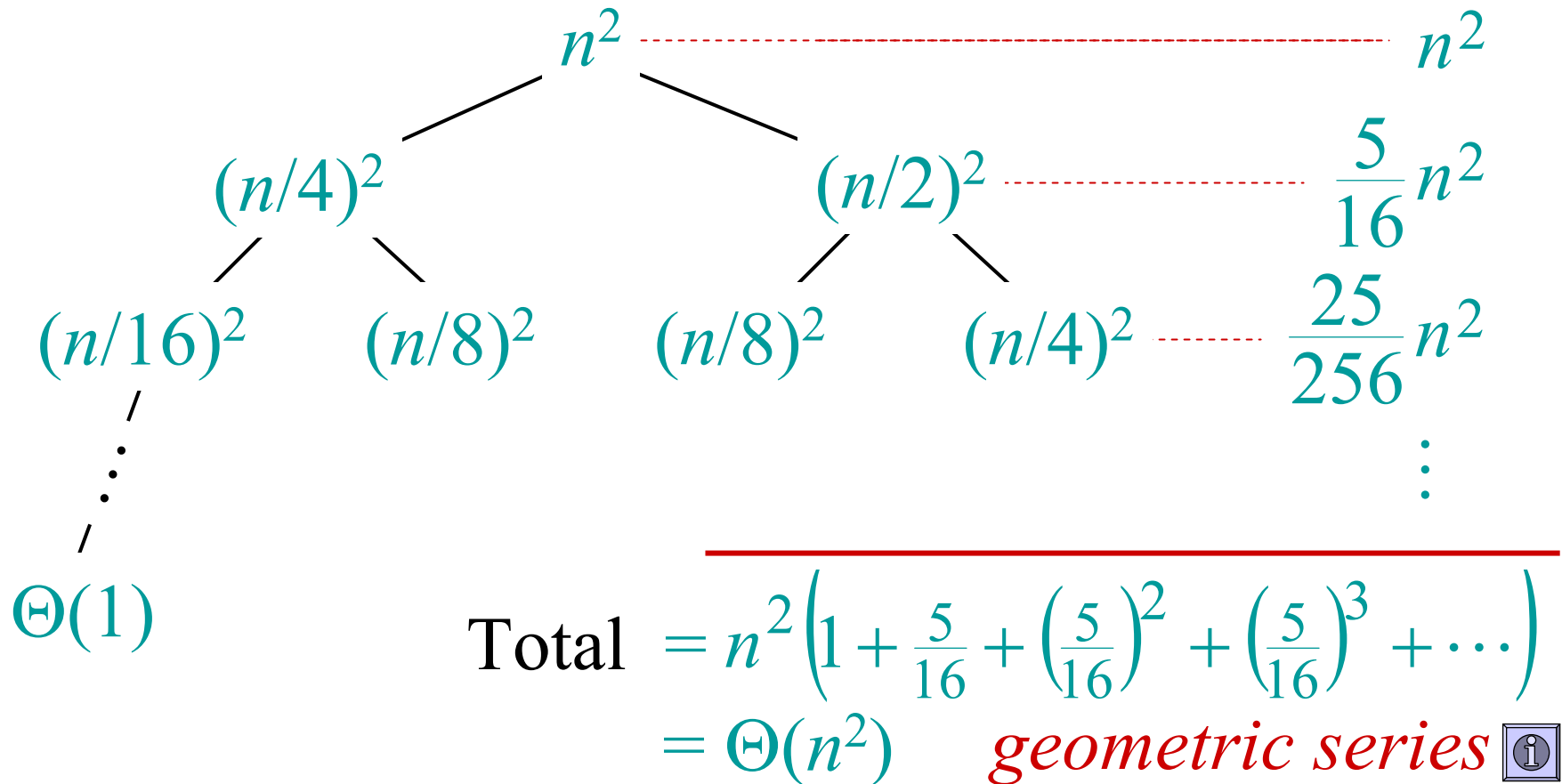


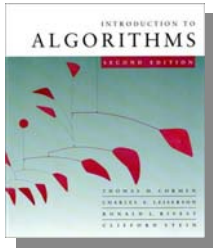




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Solve  $T(n) = T(n/4) + T(n/2) + n^2$ :



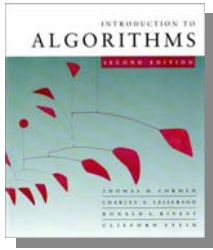


# The master method

The master method applies to recurrences of the form

$$T(n) = aT(n/b) + f(n),$$

where  $a \geq 1$ ,  $b > 1$ , and  $f$  is asymptotically positive.



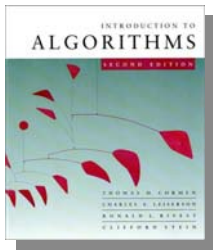
# Three common cases

Compare  $f(n)$  with  $n^{\log_b a}$ :

1.  $f(n) = O(n^{\log_b a - \varepsilon})$  for some constant  $\varepsilon > 0$ .

- $f(n)$  grows polynomially slower than  $n^{\log_b a}$  (by an  $n^\varepsilon$  factor).

**Solution:**  $T(n) = \Theta(n^{\log_b a})$ .



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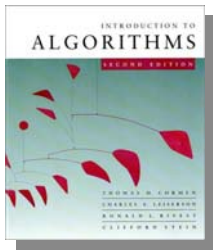
- $f(n)$  grows polynomially slower than  $n^{\log_b a}$  (by an  $n^\varepsilon$  factor).

**Solution:**  $T(n) = \Theta(n^{\log_b a})$ .

2.  $f(n) = \Theta(n^{\log_b a} \lg^k n)$  for some constant  $k \geq 0$ .

- $f(n)$  and  $n^{\log_b a}$  grow at similar rates.

**Solution:**  $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$ .



# Three common cases (cont.)

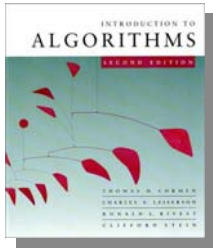
Compare  $f(n)$  with  $n^{\log_b a}$ :

3.  $f(n) = \Omega(n^{\log_b a + \varepsilon})$  for some constant  $\varepsilon > 0$ .

- $f(n)$  grows polynomially faster than  $n^{\log_b a}$  (by an  $n^\varepsilon$  factor),

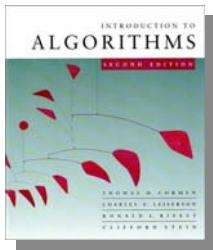
*and*  $f(n)$  satisfies the **regularity condition** that  $a f(n/b) \leq c f(n)$  for some constant  $c < 1$ .

**Solution:**  $T(n) = \Theta(f(n))$ .



# Examples

**Ex.**  $T(n) = 4T(n/2) + n$   
 $a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n.$   
**CASE 1:**  $f(n) = O(n^{2 - \varepsilon})$  for  $\varepsilon = 1.$   
 $\therefore T(n) = \Theta(n^2).$



# Examples

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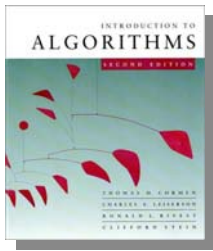
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**Ex.**  $T(n) = 4T(n/2) + n^2$

$$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^2.$$

**CASE 2:**  $f(n) = \Theta(n^2 \lg^0 n)$ , that is,  $k = 0$ .

$$\therefore T(n) = \Theta(n^2 \lg n).$$



# Examples

**Ex.**  $T(n) = 4T(n/2) + n^3$

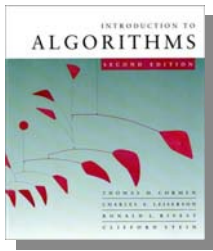
$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^3.$

**CASE 3:**  $f(n) = \Omega(n^{2 + \epsilon})$  for  $\epsilon = 1$

*and*  $4(n/2)^3 \leq cn^3$  (reg. cond.) for  $c = 1/2.$

$\therefore T(n) = \Theta(n^3).$





# Examples

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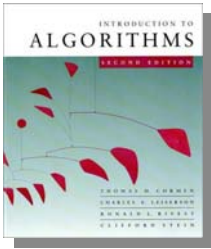
*and*  $4(n/2)^3 \leq cn^3$  (reg. cond.) for  $c = 1/2$ .

$$\therefore T(n) = \Theta(n^3).$$

**Ex.**  $T(n) = 4T(n/2) + n^2/\lg n$

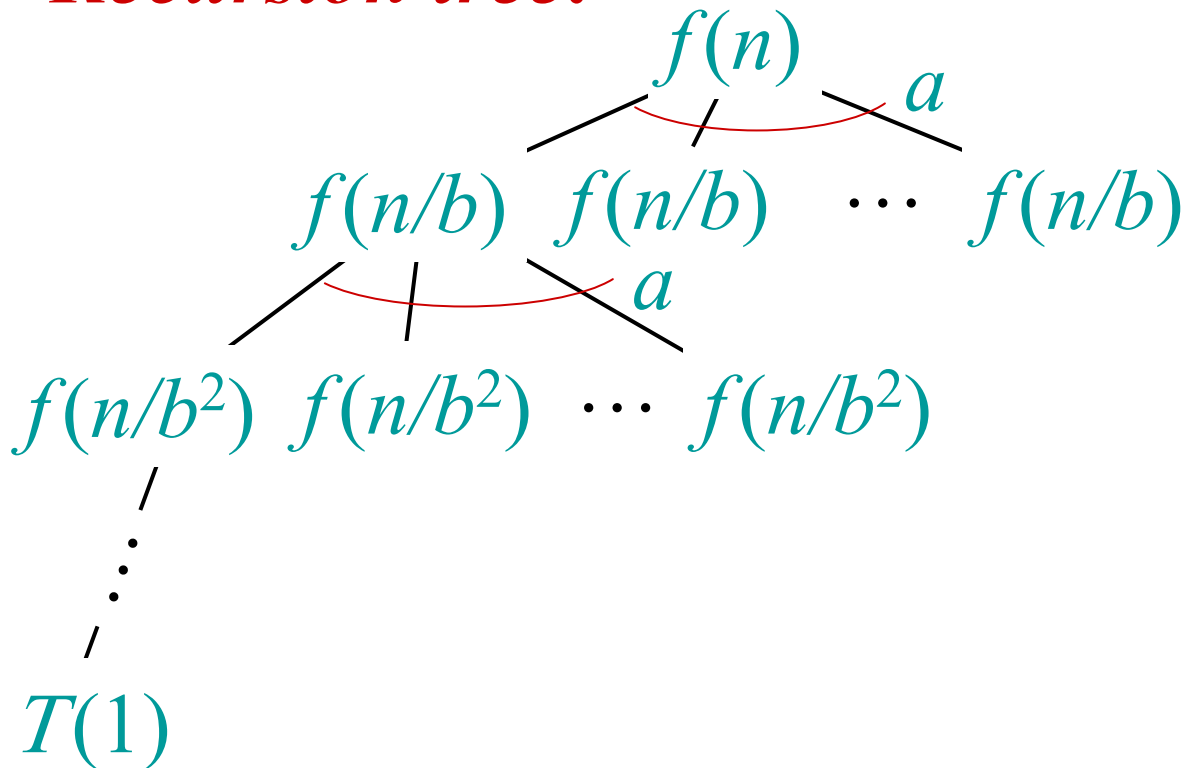
$$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^2/\lg n.$$

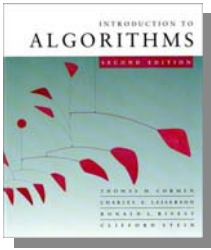
Master method does not apply. In particular, for every constant  $\varepsilon > 0$ , we have  $n^\varepsilon = \omega(\lg n)$ .



# Idea of master theorem

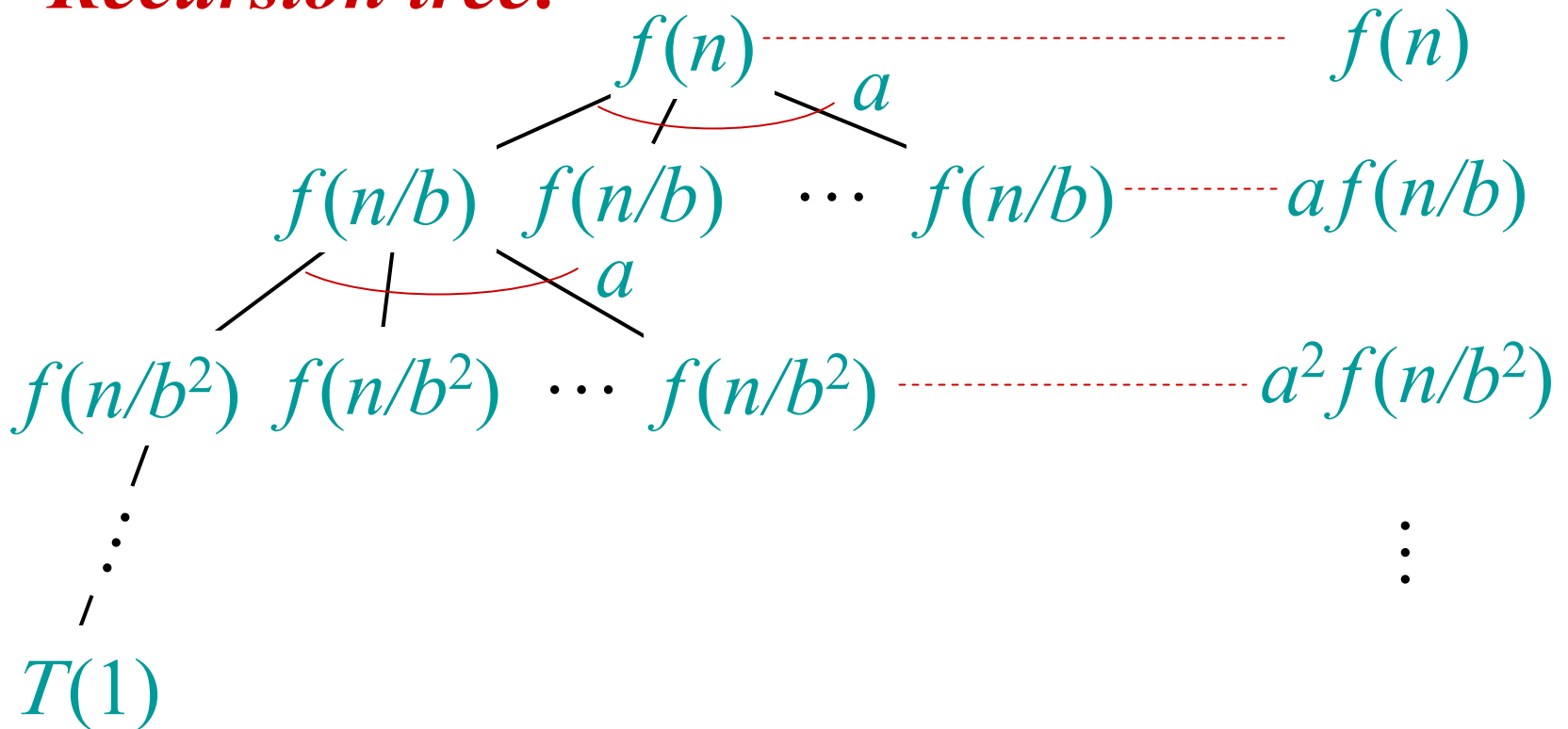
*Recursion tree:*

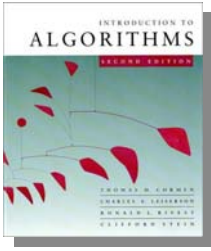




# Idea of master theorem

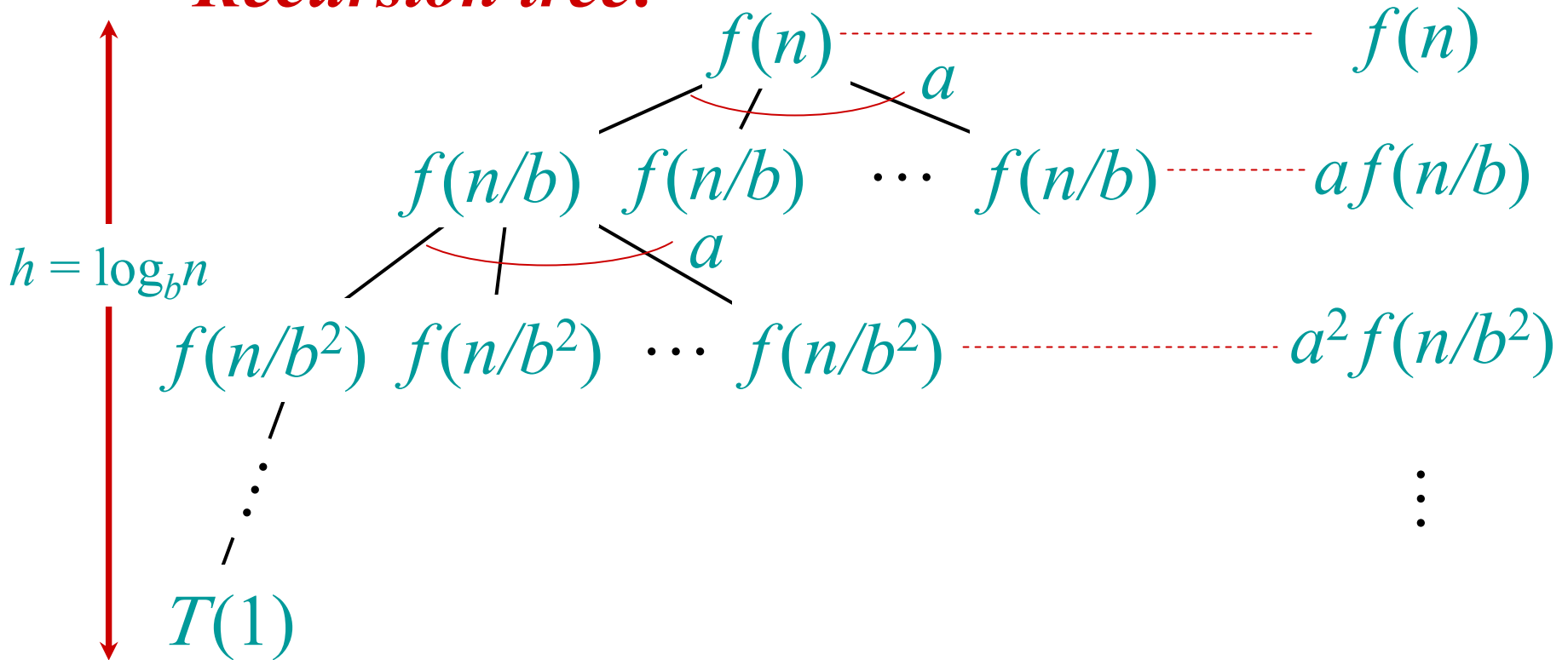
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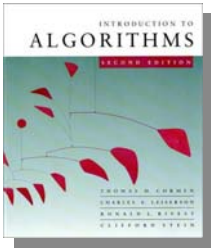




# Idea of master theorem

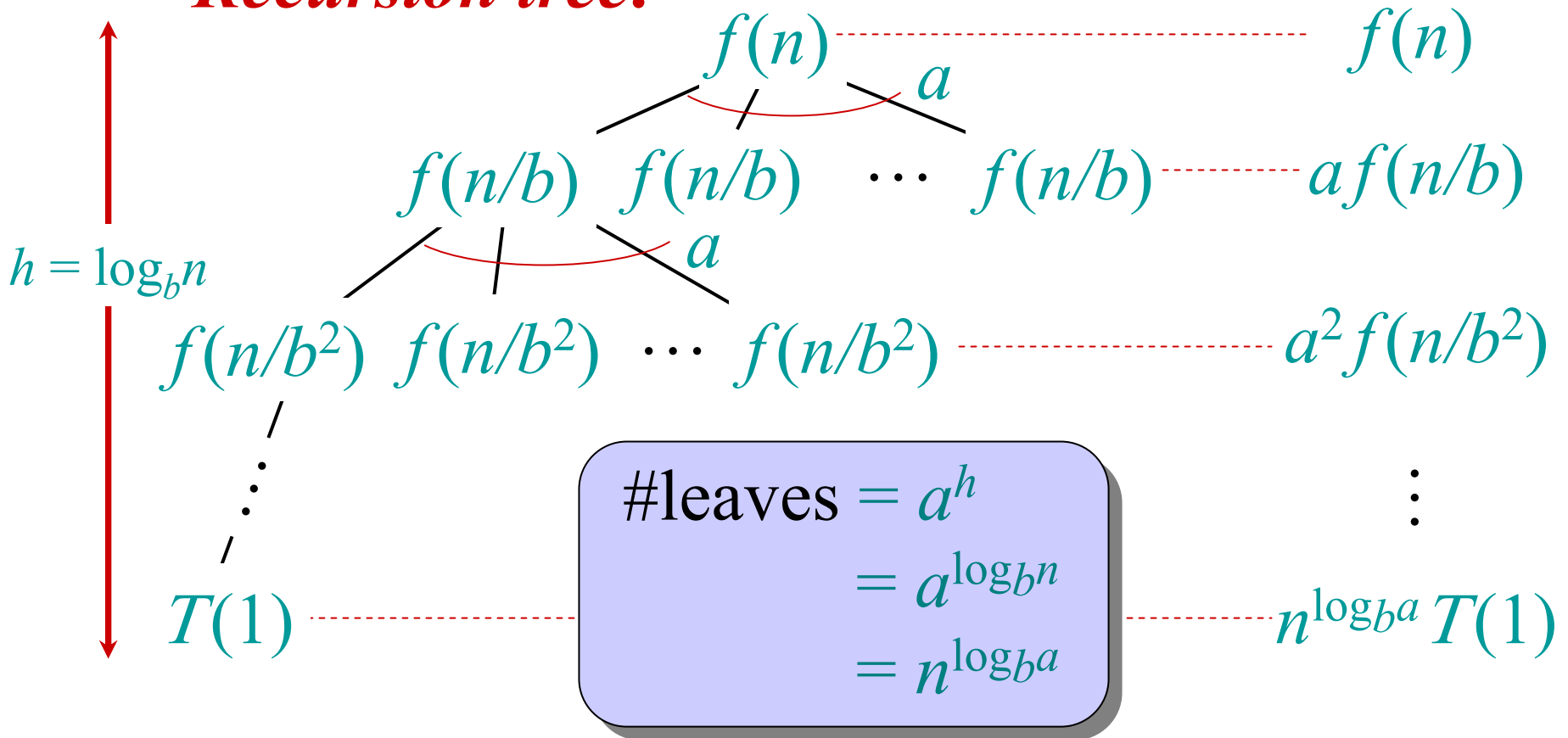
*Recursion tree:*

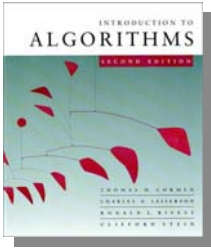




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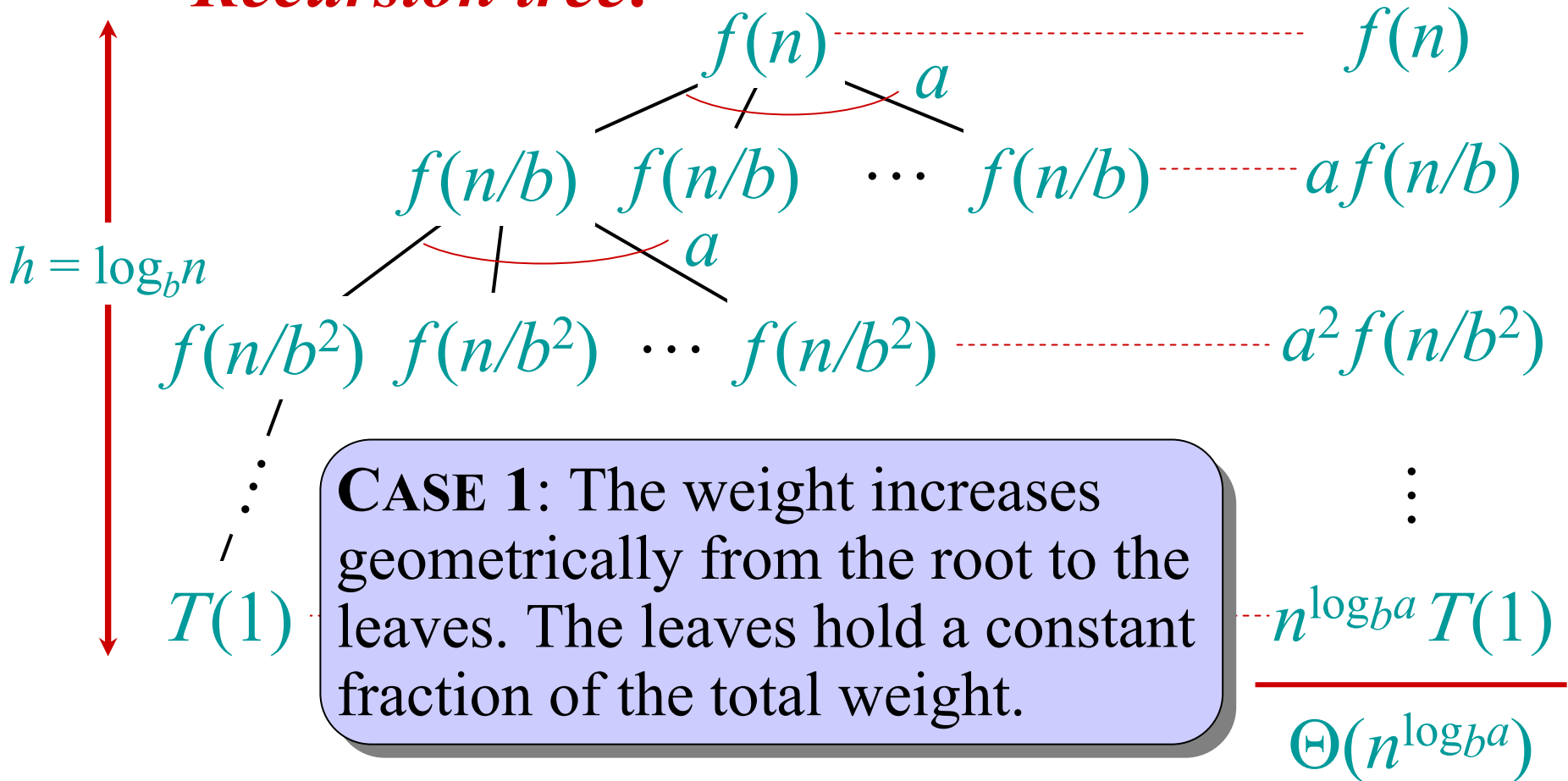
*Recursion tree:*

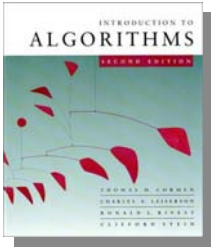




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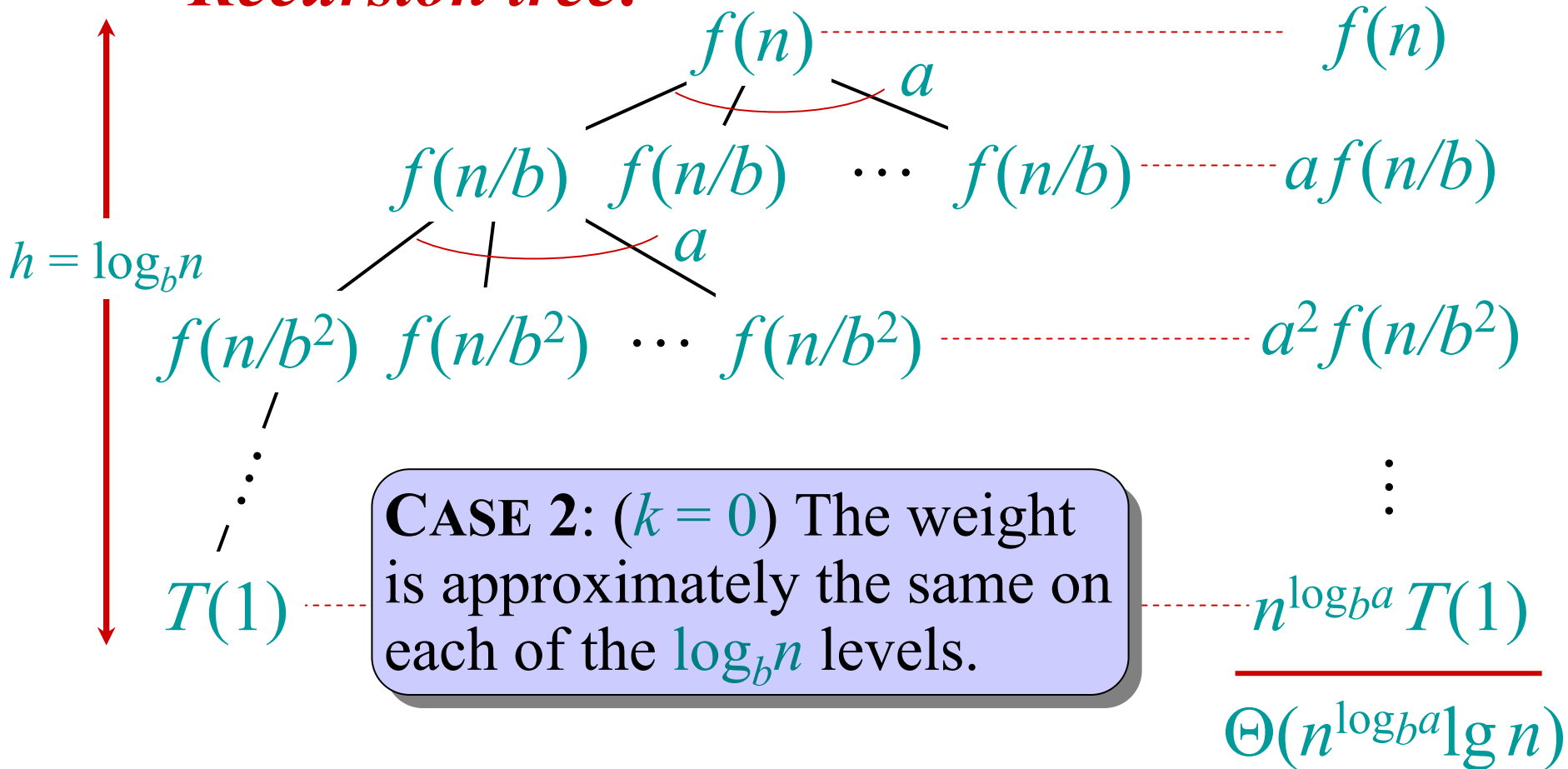
*Recursion tree:*

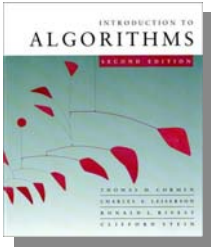




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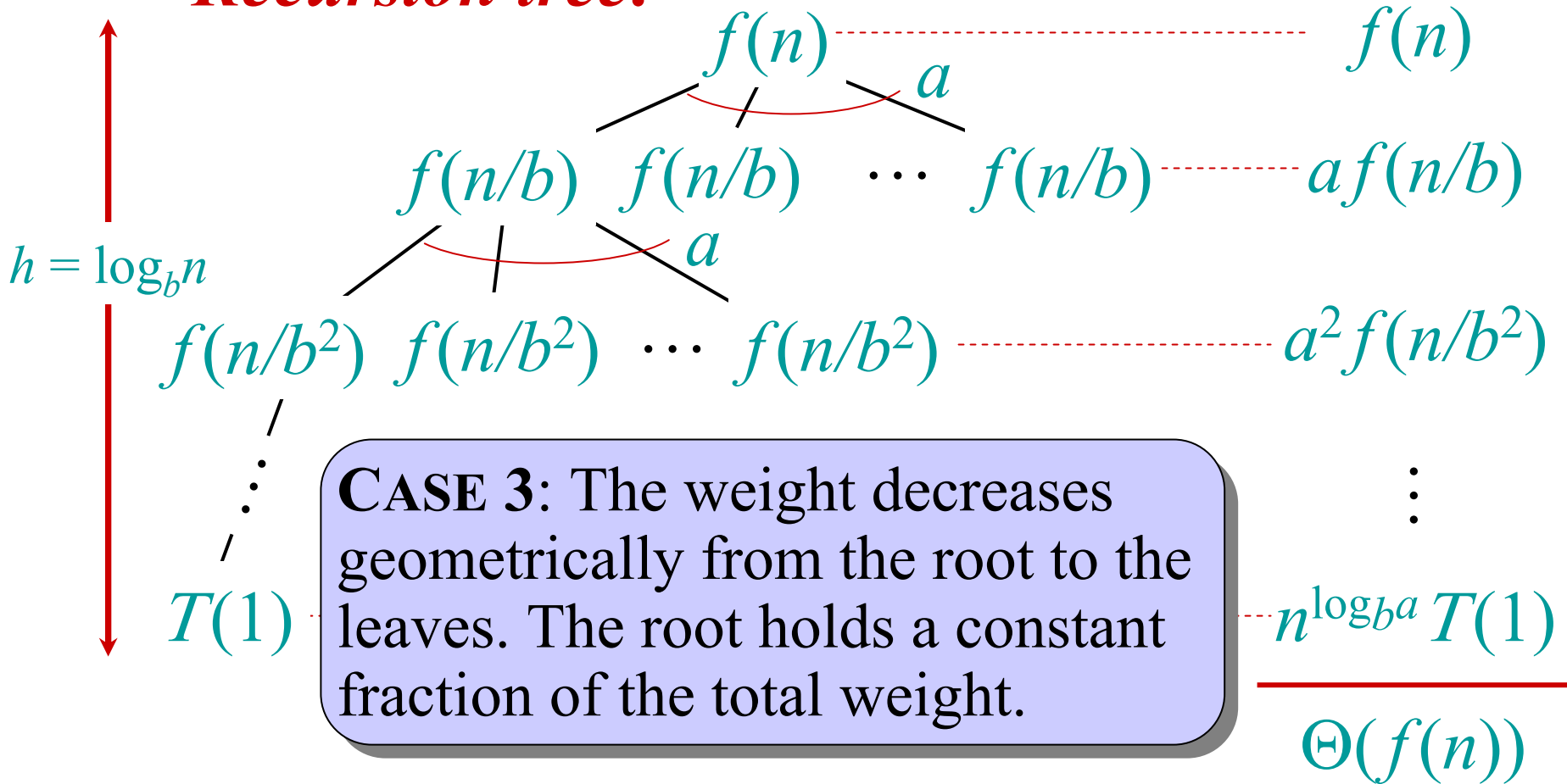
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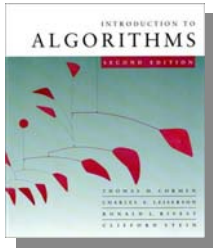


# Idea of master theorem

*Recursion tree:*







# Appendix: geometric series

$$1 + x + x^2 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad \text{for } x \neq 1$$

$$1 + x + x^2 + \dots = \frac{1}{1 - x} \quad \text{for } |x| < 1$$

Return to last  
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