# 16.30/31, Fall 2010 — Recitation # 13

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In this recitation, we tie the ideas of Lyapunov stability analysis (LSA) back to previous ways we have demonstrated stability - but also show that LSA can prove stability in cases where linear stability analysis cannot.

1 Mass-Spring-Damper, Linear Spring

Consider one of the simplest second-order systems, the mass-spring-damper system:
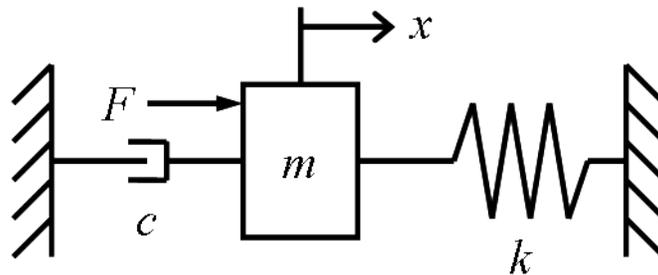
All motion is in the horizontal direction and denoted by x, where x is positive when the mass is moving to the right. The mass has a. . . , well, mass of $m > 0$. It is hooked to a damper with damping coefficient $c > 0$; it exerts a force of $-c\dot{x}$. It is also hooked to a linear spring with spring constant $k > 0$; it exerts a force of $-kx$. Finally, a constant force $F > 0$ is applied to the mass in the $+x$ direction; note that is a constant force, *not* a system input. Thus this this is an autonmous system, of the form $\dot{\mathbf{x}} = f(\mathbf{x})$ for some choice of $\mathbf{x}$.

Recall from your early statics and dynamics courses that you can derive the system dynamics by summing the forces and setting $\sum F = ma$:

$$m\ddot{x} = -c\dot{x} - kx + F$$
$$\Rightarrow \ F = m\ddot{x} + c\dot{x} + kx$$

We now get to the point of this recitation: **prove that this is a stable system.** This is an obvious conclusion, but there are many, many ways to prove it - and part of the point of this recitation is to tie these methods together. Here are just some of the ways we can do that:

1. Solve for the trajectory $x(t)$, using the above differential equation, and show that it converges to some equilibrium.
2. Derive a linearized first-order system $\dot{\mathbf{x}} = A\mathbf{x}$, and show that all eigenvalues are in the left-hand plane.
3. Use the Lyapunov stability theorem.

Let's consider each of these cases in turn.

1. The solution $x(t)$ to this second-order differential equation is the sum of the *homogeneous* solution and the *particular* solution:

$$x(t) = x_h(t) + x_p(t).$$

For the homogeneous solution, we solve the differential equation $m\ddot{x} + c\dot{x} + kx = 0$ by finding the roots of the characteristic equation:

$$ms^2 + cs + k = 0$$

$$s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.$$

Looking at the numerator, since $4mk > 0$, the square root term will either be imaginary, or real with magnitude less than c. Either way, both roots s_1 and s_2 have a negative real part. We write the homogeneous solution as

$$x_h(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t},$$

where c_1 and c_2 are unknown constants. (If $c_1 = c_2$, as is the case when $c^2 = 4mk$, the form of $x_h(t)$ will look slightly different. For the sake of brevity, we don't include that here, since the ultimate result does not change.)

For the particular solution, since the right-hand side is a constant, we see whether a solution of the form $x_p(t) = C$ satisfies the differential equation:

$$m\ddot{x}_p + c\dot{x}_p + kx_p = F$$

$$0 + 0 + kC = F \quad \Rightarrow \quad C = \frac{F}{k}$$

Thus, our general solution to the differential equation is

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + F/k.$$

We could solve for c_1 and c_2, but it is not necessary to complete our analysis. Recalling that both s_1 and s_2 had a negative real part, we know that

$$\lim_{t \to \infty} x(t) = 0 + 0 + F/k = F/k.$$

Thus we know that $x = F/k$ (which implies $\dot{x} = 0$) is a stable equilibrium. This is the displacement of the mass where the force F exactly balances the force exerted by the linear spring.

2. Let's write this second-order differential equation as a system of first-order differential equations:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -(k/m)x_1 - (c/m)x_2 + F/m.$$

We want to derive a linearized first-order system $\dot{\mathbf{x}} = A\mathbf{x}$, where $\mathbf{x} = (x_1, x_2)$. As we noted above, this system is not quite linear yet, due to the F/m term. First, we need to find the equilibrium state:

$$\begin{cases} 0 &= x_2 \\ 0 &= -(k/m)x_1 - (c/m)x_2 + F/m \end{cases}$$
$$\Rightarrow 0 = -(k/m)x_1 + F/m$$
$$\Rightarrow x_1 = F/k.$$

So our equilibrium state is $\bar{\mathbf{x}} = (F/k, 0)$. Now, perform the linearization:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \mathbf{x}.$$

(Note that we didn't actually have to plug in the value of the operating condition $\bar{\mathbf{x}}$.) Our objective is to use Lyapunov's indirect method: if the eigenvalues of the A matrix above are all in the (strict) left-hand plane, then the linearized system is stable, and thus the original nonlinear system is stable, too. The eigenvalues are found by solving $\det(sI - A) = 0$:

$$\det \begin{bmatrix} s & -1 \\ k/m & s + c/m \end{bmatrix} = s^2 + \frac{c}{m}s - \frac{k}{m} = 0$$

$$\Rightarrow ms^2 + cs + k = 0.$$

As expected, this is the same characteristic equation as before. We already showed that both roots have a negative real part, and thus are in the left-hand plane. Thus, by Lyapunov's indirect method, the system is stable about the equilibrium $(F/k, 0)$.

3. Now, let's use the Lyapunov Stability Theorem to prove that this system is stable in the sense of Lyapunov. To show this, we must pick a candidate Lyapunov function $V(\mathbf{x})$, where $\mathbf{x} = (x, \dot{x})$, which satisfies the following properties:
- $V(\mathbf{x}) \geq 0 \ \forall \mathbf{x}$
- $V(\mathbf{x}) = 0$ iff $\mathbf{x} = \bar{\mathbf{x}}$
- For all system trajectories $\mathbf{x}(t)$, $\dot{V}(\mathbf{x}(t)) \leq 0$

Above, $\bar{\mathbf{x}}$ denotes the equilibrium condition. Let's simplify things by solving for the equilibrium condition and making that the origin:

$$m(0) + c(0) + k\bar{x} = F \quad \Rightarrow \quad \bar{x} = F/k$$

Make the substitution $y = x - F/k$ in the dynamics:

$$m\ddot{y} + c\dot{y} + ky + k(F/k) = F$$

$$\Rightarrow m\ddot{y} + c\dot{y} + ky = 0.$$

What should be our Lyapunov function? Since these dynamics model a physical system (the mass-spring-damper), we expect that the system energy will decrease over time to zero. Thus, a logical choice for the Lyapunov function would be the system energy:

$$V(\mathbf{y}) = \underbrace{\frac{1}{2}m\dot{y}^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}ky^2}_{\text{spring potential energy}}$$

We now need to check whether this candidate function satisfies the properties of the Lyapunov stability theorem. As the sum of two quadratics with positive coefficients, V clearly cannot be negative (first condition). Furthermore, the only way it could be 0 is if the input to each quadratic is 0, i.e. $\dot{y} = 0$ and $y = 0$. This is at the equilibrium (second condition). Finally, let's compute the time derivative along the system trajectories:

$$\dot{V}(\mathbf{y}) = \frac{d}{dt}\left(\frac{1}{2}m\dot{y}^2 + \frac{1}{2}ky^2\right)$$

$$= m\dot{y}\ddot{y} + ky\dot{y}$$

$$= \dot{y}(m\ddot{y} + ky)$$

$$= \dot{y}(-c\dot{y})$$

$$= -c\dot{y}^2.$$

This last expression is a quadratic with a negative coefficient, which can never be positive (third condition). Thus, by the Lyapunov stability theorem, the system is stable about the origin $y = 0$ (i.e. a displacement of $x = F/k$).

Alternatively, since the dynamics in y are linear, $\dot{\mathbf{y}} = A\mathbf{y}$, we could have used a Lyapunov function $V(\mathbf{y}) = \mathbf{y}^T P \mathbf{y}$, where

$$A^T P + PA = -Q \quad (Q > 0).$$

2 Mass-Spring-Damper, Nonlinear Spring

In the mass-spring-damper system, instead of applying the force $F > 0$, suppose that the spring is *nonlinear*, exerting a force of $-k(x) = -kx^3$ for some spring constant $k > 0$. In this case, the system dynamics become

$$m\ddot{x} + c\dot{x} + kx^3 = 0.$$

From physical intuition, this should still be a stable system, but how do we prove it?

Let's again consider the three solution techniques from the first section.

1. We actually cannot use this technique, since the dynamics are no longer linear in x.
2. Our first-order nonlinear system is now

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -(k/m)x_1^3 - (c/m)x_2.$$

By solving, we can find that the equilibrium is located at $(x_1, x_2) = (0, 0)$. Perform the linearization and plug in this operating condition:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -c/m \end{bmatrix} \mathbf{x}.$$

Solve for the eigenvalues:

$$\det(sI - A) = s(s + c/m) = 0$$

Since one of the eigenvalues is zero, the linearized system is only *marginally* stable. Thus Lyapunov's indirect method *cannot* conclude anything about the stability of the nonlinear system.

3. The origin can be clearly identified as an equilibrium for this system; how do we prove it is stable via Lyapunov stability analyis? The system energy is still a good Lyapunov function candidate; however, note that the spring potential energy will take a different form here. Note that, in general, the spring potential energy is

$$PE(y) = \int_0^y k(t)dt.$$

Thus, our Lyapunov function candidate takes the form

$$V(\mathbf{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{4}kx^4.$$

It should be clear that this is a positive definite function about the equilibrium at the origin, satisfying the first two conditions of the Lyapunov stability theorem. For the third, we again look at the time derivative of V along the system trajectories:

$$\dot{V}(\mathbf{x}) = \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{4}kx^4\right)$$
$$= m\dot{x}\ddot{x} + kx^3\dot{x}$$
$$= \dot{x}(m\ddot{x} + kx^3)$$
$$= -c\dot{x}^2 \leq 0.$$

Thus, by the Lyapunov stability theorem, the origin is still a stable equilibrium with this cubic spring. Note that this was the only one of the three methods considered in this recitation that could prove stability here.

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