16.30/31, Fall 2010 — Recitation # 7

Brandon Luders

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In this recitation, we consider controllability and observability, as well as the assumptions on the LQR formulation and how to solve it by hand.

1   Controllability and Observability

Consider the following state space model:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$A = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}.$$

We are interested in determining whether this model is controllable and/or observable.

First, let's compute the transfer function matrix; since there are two inputs and two outputs, we expect the transfer function matrix to be a 2×2 matrix.

$$
\begin{aligned}
G(s) &= C(sI - A)^{-1}B \\
&= \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} s & -1 \\ 8 & s+6 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \left(\frac{1}{s(s+6)+8} \begin{bmatrix} s+6 & 1 \\ -8 & s \end{bmatrix} \right) \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{s^2 + 6s + 8} \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} s+6 & 1 \\ -8 & s \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{(s+2)(s+4)} \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ s & s \end{bmatrix} \\
&= \frac{1}{(s+2)(s+4)} \begin{bmatrix} s+2 & s+2 \\ -2(s+2) & -2(s+2) \end{bmatrix} \\
&= \frac{1}{s+4} \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}.
\end{aligned}
$$

Note the pole-zero cancellation that took place at $s = -2$; we should expect this to show up as a loss of either controllability or observability.

First, let's compute the controllability matrix $\mathcal{M}_c$:

$$\mathcal{M}_c = [\, B \quad AB \,] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & -6 & -6 \end{bmatrix}.$$

For the system to be controllable, $\mathcal{M}_c$ must have full rank. Keep in mind that an $a \times b$ matrix M (where $a \neq b$) is considered to have full rank if $\mathrm{rank}(M) = \min\{a, b\}$. So all we need is for $\mathcal{M}_c$ to have rank 2 - which it does. Thus the system is controllable.

Next, let's compute the observability matrix $\mathcal{M}_o$:

$$\mathcal{M}_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -4 & -2 \\ -8 & -4 \\ 16 & 8 \end{bmatrix}.$$

For the system to be observable, $\mathcal{M}_o$ must have rank 2. Yet we can see that all rows of $\mathcal{M}_o$ are multiples of each other, and thus $\text{rank}(\mathcal{M}_o) = 1 < 2$. Thus this system is *not* observable.

We know that a pole-zero cancellation occurred at $s = -2$, so let's investigate the controllability and observability of the mode it corresponds to. We can find the left eigenvector w^T corresponding to $s = -2$ by solving the system of equations $w^T(sI - A) = 0$. Letting $w = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^T$ and plugging in $s = -2$, we have

$$\begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 8 & -2 + 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 8 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -2w_1 + 8w_2 &= 0 \\ -w_1 + 4w_2 &= 0 \end{cases}$$

$$\Rightarrow w = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

Remember that when computing eigenvectors, this resulting system of equations should *always* be rank deficient; and thus any multiple of the w given above is also valid. To analyze the controllability of the mode, we compute

$$w^T B = \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Both elements of the resulting vector are non-zero; thus the mode is controllable.

Finally, we can find the right eigenvector v corresponding to $s = -2$ by solving the system of equations $(sI - A)v = 0$. Letting $v = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$ and plugging in $s = -2$, we have

$$\begin{bmatrix} -2 & -1 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -2v_1 - v_2 &= 0 \\ 8v_1 + 4v_2 &= 0 \end{cases}$$

$$\Rightarrow v = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

To analyze the observability of the mode, we compute

$$Cv = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since this returns the zero vector, the mode is unobservable.

2 Linear Quadratic Regulator (LQR)

The linear quadratic regulator (LQR) provides an effective way to apply full-state feedback to a system such that some desired balance is achieved between the state errors and control effort. By using LQR, you no longer have to deal with where exactly to place poles; instead, by specifying weights on the states and inputs, the closed-loop poles are chosen for you, in order to minimize a specific cost function.

In recitation today, we talked about the assumptions you can apply to LQR to help you solve for the optimal gain (Section 2.2), and worked through an example by hand (Section 2.3). First, though, let's recap the LQR formulation (Section 2.1).

2.1 Formulation

Consider the system

$$\dot{x} = Ax + Bu;$$

the objective in LQR is to place the pole locations such that the closed-loop system optimizes the cost function

$$J = \int_0^\infty \left(x^T Q x + u^T R u \right) dt,$$

where Q is the weight on the state cost and R is the weight on the input cost. Alternatively, you could specify an output $z = C_z x$ for which you would like to minimize cost, rather than the full state; the term $x^T Q x$ is then replaced with $x^T C_z^T \tilde{Q} C_z x$. You then have that $Q = C_z^T \tilde{Q} C_z$. The optimal controller takes the form $u = -Kx$, where

$$K = R^{-1} B^T P;$$

P is then found by solving the algebraic Riccati equation

$$0 = A^T P + PA + Q - PBR^{-1}B^T P.$$

Note that this is a *matrix* quadratic equation.

2.2 Assumptions

To use LQR correctly, several assumptions must be satisfied. Before specifying those assumptions, a few definitions are necessary.

Definition 1. A matrix M is *positive definite* ($M \succ 0$) if $x^T M x > 0$ for all $x \neq 0$.

Definition 2. A matrix M is *positive semidefinite* ($M \succeq 0$) if $x^T M x \geq 0$ for all x.

In practice, applying the definition directly is one of the hardest ways to determine whether a matrix is positive definite or positive semidefinite. Fortunately, there are several rules you can use to simplify this process. A few are given below.

Theorem 3. The following properties are equivalent to a matrix M being positive definite:
- All eigenvalues of M are positive.
- (Sylvester's criterion) All of the *leading principal minors* are positive. What this means is for an $n \times n$ matrix: the upper-left 1×1 corner of M has a positive determinant, the upper-left 2×2 corner of M has a positive determinant, ..., and the upper-left $n \times n$ corner of M (i.e. M itself) has a positive determinant.

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Theorem 4. The following properties are equivalent to a matrix M being positive semidefinite:

- All eigenvalues of M are non-negative.
- (Sylvester's criterion) All of the leading principal minors are non-negative.

With all that out of the way, what assumptions do we need to apply to LQR?

1. The matrix Q must be positive semidefinite, i.e. $Q \succeq 0$. This assumption ensures the state is driven to 0, because otherwise a state error might actually *decrease* the cost, which is desirable for this minimization. If $z = C_z x$ is used, we typically assume that $\tilde{Q} \succ 0$.
2. The matrix R must be positive definite, i.e. $R \succ 0$. This ensures that there is always a penalty for applying control effort.
3. The solution P to the algebraic Riccati equation is always symmetric, such that $P^T = P$. This simplifies the math involved in solving for P.
4. If (A, B, C_z) is stabilizable and detectable, then the correct solution of the algebraic Riccati equation is the unique solution P for which $P \succeq 0$. Since the equation is quadratic, you will probably get several possible solutions for P, so this assumption tells you which one to pick.
5. If (A, B, C_z) is *also* observable, then $P \succ 0$.

2.3 Example

The LQR problem we considered in recitation ended up being too messy to really solve by hand, so we consider a simpler problem below. Note that in general, solving even a simple LQR problem by hand can be hard for a 2-state system, and becomes exceedingly difficult for larger systems. Luckily, Matlab can give us the desired gain right away via `lqr(A,B,Q,R)`.

$$A = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = [\,1\,]$$

Note that $Q \succ 0 \Rightarrow Q \succeq 0$ and $R \succ 0$. Furthermore, (A, B) is controllable, and we can pick C_z appropriately such that (A, C_z) is observable; thus we are looking for the unique solution $P \succ 0$ to the algebraic Riccati equation.

We write out P symbolically as

$$P = \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

where a, b, and c are variables we need to solve for. Note that we've made P symmetric by construction.

First, let's write out the components of the algebraic Riccati equation one at a time:

$$A^T P + PA = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2a - 2b & -b - c \\ -b - c & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$-PBR^{-1}B^T P = -\begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{1} [\, 1 \quad 0 \,] \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$$= -\begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$$= -\begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$$

Note that each component is itself symmetric. Let's combine all three together!

$$0 = A^T P + PA + Q - PBR^{-1}B^T P = \begin{bmatrix} -2a - 2b & -b - c \\ -b - c & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$$

$$= \begin{bmatrix} -2a - 2b + 1 - a^2 & -b - c - ab \\ -b - c - ab & 1 - b^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This yields *three* unique equations with three unknowns:

$$\begin{cases} 0 &= -2a - 2b + 1 - a^2 \\ 0 &= -b - c - ab \\ 0 &= 1 - b^2 \end{cases}$$

From the third equation, we have $b = \pm 1$. Plugging into the first equation for the case of $b = 1$:

$$a^2 + 2a + (2b - 1) = a^2 + 2a + 1 = (a + 1)^2 = 0 \quad \Rightarrow \quad a = -1.$$

But this value of a violates Sylvester's criterion for positive definiteness, so it can't be correct. Instead, try $b = -1$:

$$a^2 + 2a + (2b - 1) = a^2 + 2a - 3 = (a + 3)(a - 1) = 0 \quad \Rightarrow \quad a = -3 \text{ or } 1.$$

Since we want $a > 0$, pick $a = 1$. Finally, from the second equation, we have

$$c = -b - ab = -(-1) - (1)(-1) = 1 + 1 = 2.$$

We've solved for all three variables; thus

$$P = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

It's easily verified that $P \succ 0$.

The last step is to actually solve for the gain:

$$K = R^{-1}B^T P = \frac{1}{1} [\, 1 \quad 0 \,] \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = [\, 1 \quad -1 \,].$$

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