4 TRANSIENT OSCILLATIONS IN NONLINEAR SYSTEMS

4.0 INTRODUCTION

We turn now from consideration of steady-state oscillations to the more complicated study of transient oscillations in nonlinear systems. The term *transient oscillation* is intended to imply a function of time which is oscillatory in nature and which can be described as a sinusoid with slowly changing amplitude and frequency. Any function could be described as a sinusoid with changing amplitude and frequency if fast enough, and perhaps discontinuous, changes were permitted. For the purpose of this chapter, however, the restriction to slowly changing amplitude and frequency is necessary to permit the use of the DF to characterize the performance of the nonlinear part of the system.

The analytic description of transient oscillations in nonlinear systems is a matter of very real practical importance. A common illustration of this is a feedback servo with integral compensation to permit an increase in low-frequency gain. Such a system frequently has a substantial chain of amplification in the forward path, amplifying the error signal to get the desired gain.

If one of these amplifiers in the chain saturates at a small value of system error, the resulting loss of effective system gain results in a decrease in relative stability due to the phase lag at low frequencies introduced by the integral compensation. The consequence of this in high-performance servos, such as those which position the gimbals of an inertial guidance stable platform, is a violent oscillation when the servo is first turned on and the error is not within the linear "notch." This oscillation is damped, and the servo eventually settles into linear operation within the notch, but the nonlinear transient oscillation is an important characteristic of the servo, and an analytic description of this characteristic is of practical importance to the designer.

Another illustration is the design of a feedback loop around a limit cycling system which regulates the amplitude of the limit cycle at some station. The design of the amplitude-regulating loop can be pursued in a rational manner only if one has a description of the dynamics relating a change in a system parameter, such as a forward gain, to the resulting change in the amplitude of the limit cycle. A transfer function which represents this dynamic effect will be derived in the following sections as a special case of the general study of transient oscillations in nonlinear systems.

4.1 ANALYTIC DESCRIPTION OF TRANSIENT OSCILLATIONS

Consider the oscillatory performance of nonlinear systems which may be cast in the form of a single loop system with separable linear and nonlinear parts, as shown in Fig. 4.1-1. This is the same form as that treated in the preceding chapter on steady-state oscillations as shown in Fig. 3.3-2. The further restriction that $r = 0$ must be made to permit practical solution of this problem. This does not rule out consideration of all forced responses, because many cases of common interest, such as a step-function response, can be given an equivalent description in terms of zero input with appropriate initial conditions on system variables. It is the response due to initial conditions which is calculated here. In the case of steady-state oscillations it is possible without undue labor to consider an input of the same form as that of the system output, namely, a steady-state oscillation. In this case, the inclusion of an input in the form of a transient oscillation is much more laborious, and would seem to be of little practical consequence.

The linear parts of the system of Fig. 4.1-1 are time-invariant operators, and are considered to be given originally in terms of their transfer functions. The nonlinear part is characterized in this analysis by its DF. It is this approximation which limits the changing amplitude and frequency of the transient oscillation to slow changes; we shall later return to the question of

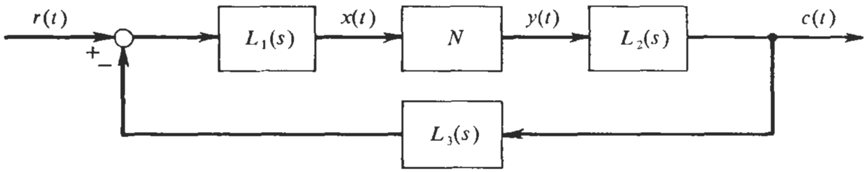

Figure 4.1-1 *Form of system considered in study of transient oscillations.*

what is meant by "slow." The use of the DF also requires the conditions on the nonlinearity and the rest of the system stated in Sec. 3.0.

DIFFERENTIAL EQUATION FOR THE NONLINEARITY INPUT

Of the variables identified in Fig. 4.1-1, the only one that can conveniently be solved for initially is x because the DF for the nonlinear part may depend on the instantaneous amplitude and frequency of the transient oscillation at x. Having found $x(t)$, it will be possible to solve for the variables at other stations, such as $c(t)$, but this will not be a simple matter. It might be noted at the outset that *in dealing with transient oscillations, not only the amplitude but the instantaneous frequency as well differ at different points around the loop.* Toward the solution for $x(t)$, we first write down the differential equation which it must satisfy. Considering the transfer functions to be rational functions of the Laplace transform variable s, we have

$$X(s) = -L_1(s)L_2(s)L_3(s)\,Y(s)$$

$$= -\frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n}\,Y(s)$$

or

$$(a_n s^n + \cdots + a_0)X(s) = -(b_0 + \cdots + b_m s^m)\,Y(s) \qquad (4.1\text{-}1)$$

This implies the differential equation

$$a_n \frac{d^n x(t)}{dt^n} + \cdots + a_0 x(t) = -\left[b_0 y(t) + \cdots + b_m \frac{d^m y(t)}{dt^m}\right] \qquad (4.1\text{-}2)$$

If the solution to this differential equation is in the form of an oscillation with slowly changing amplitude and frequency, the nonlinear relationship between $x(t)$ and $y(t)$ can profitably be approximated as the DF for the nonlinear operation $y = y(x,\dot{x})$. This approximation is employed throughout this chapter, and for notational simplicity, the relation between $x(t)$ and $y(t)$, using the DF, is written as an equality.

$$y(t) = N(t)x(t) \qquad (4.1\text{-}3)$$

Here $N(t)$ is the DF for the nonlinear part of the system. It is indicated as a function of time because it is explicitly a function of $A(t)$ and $\omega(t)$, the instantaneous amplitude and frequency of $x(t)$, both of which are functions of time.

$$a_n \frac{d^n x(t)}{dt^n} + \cdots + a_0 x(t) = -\left[b_0 N(t)x(t) + \cdots + b_m \frac{d^m N(t)x(t)}{dt^m}\right] \tag{4.1-4}$$

We now look for the solution to this equation in the form

$$x(t) = A(t) \exp [j\psi(t)] \tag{4.1-5}$$

subject to specified initial conditions $A(0)$, $\psi(0)$. Since the assumed form of the solution involves only these two constants of integration, only two initial conditions on x and its derivatives can be satisfied; $A(0)$ and $\psi(0)$ should be chosen to give $x(0)$ and $\dot{x}(0)$ correctly. The interpretation of the complex exponential form for $x(t)$ is, as always, that the physical variable $x(t)$ is either the real or imaginary part of this complex function.

To put this solution form into Eq. (4.1-4), we require the derivatives of $x(t)$ and of $N(t)x(t)$. For the terms on the left-hand side,

$$\begin{aligned} x &= A \exp (j\psi) \\ \frac{dx}{dt} &= \dot{A} \exp (j\psi) + j\dot{\psi} A \exp (j\psi) \\ &= \left(\frac{\dot{A}}{A} + j\dot{\psi}\right) A \exp (j\psi) \end{aligned} \tag{4.1-6}$$

By analogy with the standard description of exponentially decaying (or diverging) sinusoids, we define the relative rate of change of amplitude to be σ, an instantaneous exponential decay factor, and the instantaneous rate of change of phase angle to be ω.

$$\frac{\dot{A}}{A} = \sigma \tag{4.1-7}$$

$$\dot{\psi} = \omega \tag{4.1-8}$$

In this case, though, both σ and ω vary with time. Continuing the analogy further, we define a variable s as

$$s = \sigma + j\omega \tag{4.1-9}$$

In this case s has no interpretation in terms of an integral transform; it is simply a convenient variable, defined by the sequence of equations given above. In terms of this variable, Eq. (4.1-6) becomes simply

$$\frac{dx}{dt} = sx \tag{4.1-10}$$

Unfortunately, this pattern does not continue quite so simply to the higher derivatives.

$$\frac{d^2x}{dt^2} = s\dot{x} + \dot{s}x$$

$$= s^2x + \dot{s}x$$

$$= (s^2 + \dot{s})x \tag{4.1-11}$$

Similarly,

$$\frac{d^3x}{dt^3} = s^2\dot{x} + 2s\dot{s}x + \dot{s}\dot{x} + \ddot{s}x$$

$$= (s^3 + 3s\dot{s} + \ddot{s})x \tag{4.1-12}$$

Higher derivatives can be calculated as necessary. In every case, the result is expressible as x multiplied by a function of s and its derivatives.

For the terms in the right-hand member of Eq. (4.1-4),

$$\frac{d(Nx)}{dt} = N\dot{x} + \dot{N}x$$

$$= (Ns + \dot{N})x \tag{4.1-13}$$

It may be well to repeat this calculation in longer form to make clear the significance of the variables. Using the polar form for the DF,

$$N(t) = \rho_N(t) \exp [j\theta_N(t)]$$

$$Nx = \rho_N A \exp [j(\psi + \theta_N)]$$

$$\frac{d(Nx)}{dt} = \dot{\rho}_N A \exp [j(\psi + \theta_N)] + \rho_N \dot{A} \exp [j(\psi + \theta_N)]$$

$$+ j(\dot{\psi} + \dot{\theta}_N)\rho_N A \exp [j(\psi + \theta_N)]$$

$$= \left[\left(\frac{\dot{A}}{A} + j\dot{\psi}\right)\rho_N \exp (j\theta_N) + \dot{\rho}_N \exp (j\theta_N)\right.$$

$$\left. + j\dot{\theta}_N\rho_N \exp (j\theta_N)\right] A \exp (j\psi)$$

$$= (Ns + \dot{N})x$$

Similarly,

$$\frac{d^2(Nx)}{dt^2} = Ns\dot{x} + N\dot{s}x + \dot{N}sx + \dot{N}\dot{x} + \ddot{N}x$$

$$= [N(s^2 + \dot{s}) + 2\dot{N}s + \ddot{N}]x \tag{4.1-14}$$

and higher derivatives can be calculated as required.

Now these expressions for $x(t)$ and its derivatives and $N(t)x(t)$ and its derivatives can be put into Eq. (4.1-4), and the factorable $x(t)$ divided out of every term; that factor represents the trivial solution to the equation. The result is an equation among s and its derivatives and N and its derivatives. Since s and N are functions of A, $\dot{A}$, and $\omega = \dot{\psi}$, the complex equation, if written out in terms of its real and imaginary parts, becomes two nonlinear differential equations in A, ψ, and their derivatives. The solution to these equations defines $A(t)$ and $\psi(t)$, and thus the system variable $x(t)$, through Eq. (4.1-5). These differential equations, being nonlinear, generally do not permit an explicit solution. However, the purpose in deriving differential equations for A and ψ using an assumed solution form was to take advantage of the slowly changing nature of the parameters A and ω to effect useful approximate solutions.

REPLACEMENT RULES FOR TRANSIENT OSCILLATIONS

We shall return to the solution of the system equation, but let us first review the procedure involved in setting up that equation. Having once calculated the derivatives of the important variables, it is possible to go directly from the transfer function for the linear part of the system to the differential equation for A and ψ. From the block diagram of the system and the open-loop transfer function, one can immediately write an equation of the form of Eq. (4.1-1).

$$(a_n s^n + \cdots + a_1 s^1 + a_0 s^0)X(s) = -(b_0 s^0 + b_1 s^1 + \cdots + b_m s^m)Y(s) \tag{4.1-15}$$

The equation defining the transient oscillation is then derived from this by a few steps which can be formalized into simple rules:

1. Cross out $X(s)$ on the left, and $Y(s)$ on the right.
2. On the left-hand side, replace

s^0 by 1

s^1 by s

s^2 by $s^2 + \dot{s}$

s^3 by $s^3 + 3s\dot{s} + \ddot{s}$

s^4 by $s^4 + 6s^2\dot{s} + 3\dot{s}^2 + 4s\ddot{s} + \dddot{s}$

s^5 by $s^5 + 10s^3\dot{s} + 15s\dot{s}^2 + 10\dot{s}\ddot{s} + 10s^2\ddot{s} + 5s\dddot{s} + \ddddot{s}$

. . . .

3. On the right-hand side, replace

s^0 by N

s^1 by $Ns + \dot{N}$

s^2 by $N(s^2 + \dot{s}) + 2\dot{N}s + \ddot{N}$

s^3 by $N(s^3 + 3s\dot{s} + \ddot{s}) + 3\dot{N}(s^2 + \dot{s}) + 3\ddot{N}s + \dddot{N}$

. . . .

Replacement rules 2 and 3 can be expressed in compact form in terms of recursion formulas.[1] Let rule 2 read: "On the left-hand side, replace s^n by f_n," and rule 3: "On the right-hand side, replace s^n by g_n." These replacement functions are given by

$$f_{n+1} = \sum_{k=0}^{n} \binom{n}{k} (p^{n-k}s) f_k \qquad f_0 = 1 \qquad (4.1\text{-}16)$$

$$g_{n+1} = \sum_{k=0}^{n} \binom{n}{k} [(p^{n-k}s) g_k + (p^{n+1-k}N) f_k] \qquad g_0 = N \qquad (4.1\text{-}17)$$

where

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

and p^n represents d^n/dt^n.

The result of the application of these rules is a differential equation involving complex terms. One then writes two equations involving only real terms by equating real and imaginary parts of the original equation. This involves the expansion of terms like

$$s^2 = \sigma^2 - \omega^2 + j2\sigma\omega$$

$$s^3 = \sigma^3 - 3\sigma\omega^2 + j(3\sigma^2\omega - \omega^3)$$

and the writing of derivatives like

$$\dot{s} = \dot{\sigma} + j\dot{\omega}$$

$$\ddot{s} = \ddot{\sigma} + j\ddot{\omega}$$

The derivatives of N are more complicated. Using the real and imaginary form for $N(A,\omega)$, we have

$$N(A,\omega) = n_p(A,\omega) + jn_q(A,\omega)$$

$$\dot{N} = \frac{\partial n_p}{\partial A}\dot{A} + \frac{\partial n_p}{\partial \omega}\dot{\omega} + j\frac{\partial n_q}{\partial A}\dot{A} + j\frac{\partial n_q}{\partial \omega}\dot{\omega}$$

$$= A\frac{\partial n_p}{\partial A}\sigma + \frac{\partial n_p}{\partial \omega}\dot{\omega} + j\left(A\frac{\partial n_q}{\partial A}\sigma + \frac{\partial n_q}{\partial \omega}\dot{\omega}\right)$$

[1] Pointed out by B. C. Sherman.

The higher derivatives of N involve higher-ordered partial derivatives of n_p and n_q with respect to A and ω, as well as higher time derivatives of σ and ω. We note that in the quasi-static situation in which derivatives of s and N are ignored, the substitution rules reduce to identities; i.e., application of the rules reproduces the original system transfer function.

Example 4.1-1 As an example of the application of these rules, consider the system of Fig. 4.1-2. This simple system has an ideal relay driving a second-order plant in a closed-loop configuration. For this system, the equation corresponding to Eq. (4.1-15) is

$$(s^2 + bs)X(s) = -KY(s) \tag{4.1-18}$$

Application of the rules 1 to 3 gives

$$s^2 + \dot{s} + bs = -KN \tag{4.1-19}$$

as the equation defining the transient oscillation. Real and imaginary parts of this equation are

Real: $$\sigma^2 - \omega^2 + \dot{\sigma} + b\sigma = -Kn_p \tag{4.1-20a}$$

Imaginary: $$2\sigma\omega + \dot{\omega} + b\omega = -Kn_q \tag{4.1-20b}$$

Using the DF for the ideal relay, these become

Real: $$\sigma^2 - \omega^2 + \dot{\sigma} + b\sigma + \frac{4KD}{\pi A} = 0 \tag{4.1-21a}$$

Imaginary: $$2\sigma\omega + \dot{\omega} + b\omega = 0 \tag{4.1-21b}$$

These equations will be solved in Sec. 4.3.

SOLUTION FOR OTHER VARIABLES

In most cases, the solution for $x(t)$ serves adequately to describe the dynamic character of an oscillatory system. In some situations, however, one might wish to determine the response at some other station, such as $c(t)$ as shown in Fig. 4.1-1. From that figure one can write two relations which determine $c(t)$ in terms of $x(t)$.

$$C(s) = L_2(s)NX(s) \tag{4.1-22}$$

$$C(s) = -[L_1(s)L_3(s)]^{-1}X(s) \tag{4.1-23}$$

Each of these implies a differential equation of the form

$$g_n \frac{d^n c(t)}{dt^n} + \cdots + g_0 c(t) = f(t) \tag{4.1-24}$$

where the right-hand member is a known function of time composed of a linear combination of $N(t)x(t)$ and its derivatives if Eq. (4.1-22) is used, or a linear combination of $x(t)$ and its derivatives if Eq. (4.1-23) is used. In

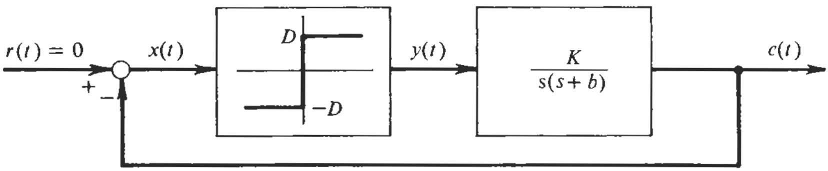

Figure 4.1-2 *System of Example 4.1-1.*

either case, $x(t)$, and consequently $N(t)$, are determined, together with their derivatives, by the solutions to the equations corresponding to Eq. (4.1-21); so $f(t)$ in Eq. (4.1-24) is determined. The choice between use of Eq. (4.1-22) or (4.1-23) is purely one of convenience, and must be decided in every individual case. For example, if $L_1(s)$ and $L_3(s)$ have no zeros, Eq. (4.1-24) takes the particularly simple form

$$c(t) = \frac{1}{g_0} f(t) \tag{4.1-25}$$

which can be handled readily. In more complicated cases this is still no more than the problem of calculating the output of a linear filter due to a specified time function input. Both exact and approximate methods of treating this problem are discussed in most books on control theory. (See, for example, Ref. 9, secs. 1.6, 6.6.)

The historical development of the analytical approach presented in this section is discussed in the next; then, in Sec. 4.3, we consider the solution of the equations defining a transient oscillation.

4.2 RELATION TO OTHER WORK

The solution of differential equations by means of fitting a sinusoid with slowly varying amplitude and phase angle is associated with a number of people, but foremost among them are Krylov and Bogoliubov, as the result of their extensive work in the period 1930 to 1940. Much of their attention was directed toward the discovery of stationary solutions corresponding to steady-state oscillations, and the stability of these solutions, but their approach was by way of the more general transient oscillation.

This work was cited as a beginning point in Chap. 2. The pertinent results presented there are that, for the differential equation

$$\ddot{x} + \omega_0^2 x + \mu f(x,\dot{x}) = 0 \tag{4.2-1}$$

Krylov and Bogoliubov find as their solution of first approximation

$$x(t) = A(t) \sin [\omega_0 t + \theta(t)] \tag{4.2-2}$$

where $A(t)$ and $\theta(t)$ satisfy the differential relations

$$\frac{dA}{dt} = -\frac{\mu}{2\pi\omega_0}\int_0^{2\pi} f(A\sin\psi, A\omega_0\cos\psi)\cos\psi\, d\psi \tag{4.2-3}$$

and

$$\frac{d\theta}{dt} = \frac{\mu}{2\pi A\omega_0}\int_0^{2\pi} f(A\sin\psi, A\omega_0\cos\psi)\sin\psi\, d\psi \tag{4.2-4}$$

The total rate of change of phase, ω, is

$$\omega = \omega_0 + \frac{\mu}{2\pi A\omega_0}\int_0^{2\pi} f(A\sin\psi, A\omega_0\cos\psi)\sin\psi\, d\psi \tag{4.2-5}$$

In the notation of the preceding section, these results are

$$\sigma = \frac{\dot{A}}{A} = -\frac{n_q(A,\omega_0)}{2\omega_0} \tag{4.2-6}$$

$$\omega = \omega_0 + \frac{n_p(A,\omega_0)}{2\omega_0} \tag{4.2-7}$$

Application of the rules of the preceding section to Eq. (4.2-1) gives

$$s^2 + \dot{s} + \omega_0^2 + N(A,\omega) = 0 \tag{4.2-8}$$

which has real and imaginary parts

Real: $$\sigma^2 - \omega^2 + \dot{\sigma} + \omega_0^2 + n_p(A,\omega) = 0 \tag{4.2-9a}$$

Imaginary: $$2\sigma\omega + \dot{\omega} + n_q(A,\omega) = 0 \tag{4.2-9b}$$

The Krylov and Bogoliubov solution may be recognized as an approximate solution to these equations. In the equation of imaginaries, drop the $\dot{\omega}$ term and approximate ω by ω_0. This gives the solution for σ [Eq. (4.2-6)]. In the equation of reals, drop the σ^2 and $\dot{\sigma}$ terms and approximate ω by ω_0 in the evaluation of n_p. Then the neglect of n_p/ω_0^2 raised to powers greater than 1 gives the solution for ω [Eq. (4.2-7)].

This would seem to be a rather crude solution to Eqs. (4.2-9); in particular, it will be shown in examples that the omission of $\dot{\omega}$ in the equation of imaginaries is often serious. Moreover, we should like to include, in our consideration, systems for which $\omega_0 = 0$. In that case there is no evident approximation to be used for ω in evaluating n_p and n_q, and n_p/ω_0^2 cannot be considered a small quantity. We also wish to deal with systems of arbitrary order. Krylov and Bogoliubov had to consider a second-order equation of such a form that it permitted an asymptotic solution involving expansions in ascending powers of the small parameter μ to retain a degree of mathematical rigor. Under some conditions, for example, they are able to calculate bounds on the error in the solution due to the harmonic linearization of the nonlinearity.

The point of view taken in the preceding section was quite different. We began with a system of arbitrary order and, without stated motivation, employed the DF to characterize the nonlinearity, making no assumption about smallness of the nonlinear effect. This approach is actually motivated by the fact that it is the only practicable means of attacking an important class of problems; it is to some degree justified by the excellent results it yields in test problems. It is of some comfort to note, however, that after approaching the study of oscillatory nonlinear systems from a rather different point of view, Krylov and Bogoliubov were led to characterize the nonlinearity by the use of "harmonic linearization," which in the first approximation is exactly the describing function.

More recently, other writers have discussed the subject of oscillatory transients in nonlinear systems. Grensted (Ref. 5) and Voronov (Ref. 10), the latter following Popov (Ref. 7), give essentially the same argument as that of the preceding section. Neither of them, however, recognized the convenience of the variable $s = \dot{A}/A + j\dot{\psi}$ in writing out the derivatives of the oscillatory function. Clauser (Ref. 1) gives derivative formulas in the same form as those derived here, but he speaks of the result as an operational calculus for nonlinear oscillations. In this chapter we simply employ s as a convenient working variable and make no interpretation of it as a differential operator.

A number of writers have taken a quasi-static approach to the study of transient oscillations, among them Gelb and Vander Velde (Refs. 3 and 4), Lubbock and Barker (Ref. 6), and several others whose results are summarized by Thaler and Pastel (Ref. 8, sec. 4.9). In this approach one solves the standard characteristic equation of the system, using the DF to represent the nonlinearity, to determine the normal modes corresponding to prescribed values of A and ω. If the oscillatory mode, which is the one of interest, has a damping factor different from zero, this specifies the instantaneous relative rate of change of amplitude. The determination of this rate of change of amplitude as a function of the amplitude and corresponding frequency permits solution for the amplitude and frequency transients. If this solution is further approximated by considering the performance of the nonlinearity to be essentially fixed over each half-cycle of the oscillation, the transient is pieced together from half-cycles of sinusoids having a positive or negative damping factor which corresponds to the amplitude and frequency of the oscillation at the beginning of each half-cycle. This last procedure is appropriately referred to as *piecewise-sinusoidal approximation*.

There are many situations in which the quasi-static solution has a useful degree of accuracy. This is exactly the solution one obtains from the equations developed by the rules of the preceding section if all derivatives of σ and ω are dropped. Having the complete equations, including the derivative terms, one could first obtain the quasi-static solution, and from it calculate the σ and ω derivatives to check their magnitudes. It will be found in the examples of the following sections that quasi-static solutions may be appreciably in error; usually, in these cases, it is the $\dot{\omega}$ contribution which is important.

4.3 SOLUTION OF THE EQUATIONS DEFINING THE OSCILLATION

We now address the question of how to solve the nonlinear differential equations for A and ω derived in Sec. 4.1. Only in rare cases would one expect to find an exact solution; so our purpose is to find useful approximate solutions. The approximations to be made are indicated by the fact that we have formulated the problem in terms of variables A and ω, which we expect to be slowly varying. Specifically, if the DF is to approximate the performance of the nonlinearity,[1] the input to the nonlinearity, $x(t)$, must approximate a sinusoid. That is, the relative changes in amplitude and frequency of $x(t)$ over one period of the oscillation must be small.

$$\frac{|\Delta A \text{ in one period}|}{A} = \frac{|\dot{A}|}{A}\frac{2\pi}{\omega} = 2\pi\frac{|\sigma|}{\omega} \ll 1 \qquad (4.3\text{-}1)$$

$$\frac{|\Delta\omega \text{ in one period}|}{\omega} = \frac{|\dot{\omega}|}{\omega}\frac{2\pi}{\omega} = 2\pi\frac{|\dot{\omega}|}{\omega^2} \ll 1 \qquad (4.3\text{-}2)$$

With respect to ω (or ω^2), then, we should expect σ and $\dot{\omega}$ to be small, and any higher derivatives or higher powers of these quantities to be still smaller.

With this order for the importance of variables in mind, recall that the Krylov and Bogoliubov solution could be derived from our equations for A and ω by solving the equation of reals for ω and the equation of imaginaries for σ, after having dropped certain terms. Additional motivation for calculating ω from the equation of reals and σ from the equation of imaginaries is provided by considering the describing function for the nonlinear part as the sum of a proportional plus a derivative gain. The real part is the proportional gain, which acts as a "spring constant" in the oscillatory system, thus affecting the frequency of the oscillation. The imaginary part is the derivative gain, which influences the damping of the oscillation.

Following this pattern, and keeping as many terms as can conveniently be retained, we formulate a procedure for effecting a solution which will be termed the *small-σ solution*.

1. In the equation of reals, drop derivatives of σ and ω. Powers of σ greater than 2 can usually be dropped as well. Solve for $\omega(\sigma,A)$.

[1] For an interesting though considerably more complicated alternative to the DF as the descriptor for the nonlinearity, see Ref. 2.

2. Differentiate the resulting expression for ω to get $\dot{\omega}(\sigma,A)$ in which derivatives of σ are dropped.
3. In the equation of imaginaries, drop terms involving derivatives of σ and derivatives higher than the first of ω. Using $\omega(\sigma,A)$ and $\dot{\omega}(\sigma,A)$ from steps 1 and 2, solve for $\sigma(A)$. In this solution there is often little to be gained by retaining powers of σ greater than 1; powers greater than 2 should always be negligible.
4. Calculate time from the expression

$$t(A) = \int_{A(0)}^{A} \frac{1}{A\sigma(A)}\, dA$$

From this one can calculate directly $A(t)$, $\sigma(t)$, $\omega(t)$, $\psi(t)$, and thus $x(t)$.

The *quasi-static solution* results from dropping all derivative terms in σ, ω, and N from both equations. The two equations are then just algebraic equations for σ and ω, with A as a parameter, and can be solved by numerical iteration, graphing, or any other method. The solution is exactly that which one obtains by root-locus construction, using the original transfer function for the linear part of the system and treating A as a parameter for the purpose of evaluating the DF for the nonlinear part. Having this quasi-static solution for $\sigma(A)$ and $\omega(A)$, step 4 can be completed, as in the case of the small-σ solution. The quasi-static solution is mentioned primarily for the purpose of comparison.

If all derivative terms in σ, ω, and N are dropped from the differential equations for A and ω, as in the quasi-static case, and in addition σ is set equal to zero, the resulting equations are static relations in A and ω. If the solution, which we refer to as the *static*, or *steady-state*, *solution*, is nontrivial, the static values of A and ω define a steady-state oscillation which is a system limit cycle.

Example 4.3-1 We now illustrate these solution procedures in the case of the example system shown in Fig. 4.1-2. This solution can be generalized through nondimensionalization. If the nondimensional unit of time bt is employed, together with the nondimensional frequency variable

$$p = \frac{s}{b} = \frac{\sigma}{b} + j\frac{\omega}{b} = \mu + j\lambda \tag{4.3-3}$$

and the output level of the switch is associated with the gain of the linear part, the system diagram of Fig. 4.3-1 results in which there remains only a single parameter to characterize the system. That parameter, KD/b^2, is the steady-state slewing velocity of the system in terms of nondimensional time, and for simplicity will be designated V.

$$V = \frac{KD}{b^2} \tag{4.3-4}$$

The system equation, corresponding to Eq. (4.1-15), is then

$$(p^2 + p)X(p) = -VY(p) \tag{4.3-5}$$

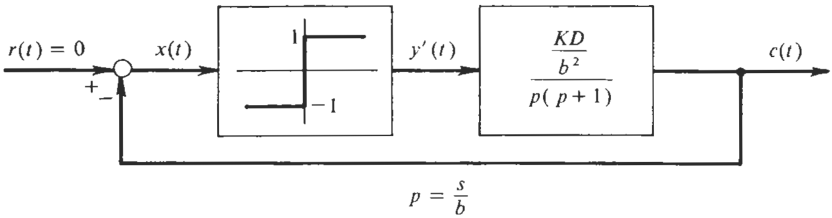

Figure 4.3-1 *Nondimensionalized system of Example 4.3-1.*

Application of the replacement rules gives

$$p^2 + \dot{p} + p = -VN \tag{4.3-6}$$

which has real and imaginary parts

Real: $$\mu^2 - \lambda^2 + \dot{\mu} + \mu + \frac{4V}{\pi A} = 0 \tag{4.3-7a}$$

Imaginary: $$2\mu\lambda + \dot{\lambda} + \lambda = 0 \tag{4.3-7b}$$

To obtain the small-σ solution, the equation of reals is written

$$\mu^2 - \lambda^2 + \mu + \frac{4V}{\pi A} = 0 \tag{4.3-8}$$

from which $$\lambda = \sqrt{\frac{4V}{\pi A} + \mu + \mu^2} \tag{4.3-9}$$

Thus $$\dot{\lambda} = \frac{-(4V/\pi A^2)\dot{A} + \dot{\mu} + 2\mu\dot{\mu}}{2\lambda} \tag{4.3-10}$$

where the dot indicates differentiation with respect to bt. According to step 2, we drop the $\dot{u}$ terms. Also noting that $\dot{A}/A = \mu$, we get

$$\dot{\lambda} = -\frac{(4V/\pi A)\mu}{2\lambda}$$

$$= -\frac{1}{2\lambda}(\mu\lambda^2 - \mu^2 - \mu^3) \tag{4.3-11}$$

The equation of imaginaries is then written according to step 3.

$$2\mu\lambda + \dot{\lambda} + \lambda = 0$$

$$2\mu + \frac{\dot{\lambda}}{\lambda} + 1 = 0$$

$$2\mu - \tfrac{1}{2}\mu + \frac{1}{2}\left(\frac{\mu}{\lambda}\right)^2(1 + \mu) + 1 = 0 \tag{4.3-12}$$

At this point it would be possible to solve Eqs. (4.3-12) and (4.3-9) simultaneously for μ and λ as functions of A, but a simpler solution is adequate. According to Eq. (4.3-1),

μ/λ must be a small quantity if the DF is to represent the nonlinearity adequately, and there is no need to retain the second-degree term in this small quantity. Thus

$$\mu = -\tfrac{2}{3} \tag{4.3-13}$$

and

$$\lambda = \sqrt{\frac{4V}{\pi A} - \frac{2}{9}} \tag{4.3-14}$$

from Eq. (4.3-9).

For a system with unity feedback, $c(t) = -x(t)$; so this solution for $x(t)$ can be interpreted immediately in terms of $c(t)$ just by using initial conditions corresponding to $c(t)$. We shall work with the real parts of the complex functions, and use $\tau = bt$ as the nondimensional time variable.

$$\begin{aligned} c(\tau) &= \operatorname{Re}\{A(\tau)\exp[j\psi(\tau)]\} \\ &= A(\tau)\cos\psi(\tau) \end{aligned} \tag{4.3-15}$$

$$\dot{c}(\tau) = \mu(\tau)A(\tau)\cos\psi(\tau) - \lambda(\tau)A(\tau)\sin\psi(\tau) \tag{4.3-16}$$

The initial conditions are determined from these expressions.

$$c(0) = A(0)\cos\psi(0) \tag{4.3-17}$$

$$\dot{c}(0) = \mu(0)A(0)\cos\psi(0) - \lambda(0)A(0)\sin\psi(0) \tag{4.3-18}$$

For the case $c(0)/V = 1$, $\dot{c}(0) = 0$, we find

$$\frac{A(0)}{V} = 1.247 \qquad \psi(0) = -36.68^\circ \tag{4.3-19}$$

Equations (4.3-19) and (4.3-13) imply

$$\frac{A(\tau)}{V} = 1.247\exp\left(-\tfrac{2}{3}\tau\right) \tag{4.3-20}$$

and Eq. (4.3-14) becomes

$$\lambda(\tau) = \sqrt{\frac{4}{1.247\pi}\exp\left[\frac{2}{3}\tau\right] - \frac{2}{9}} \tag{4.3-21}$$

Finally,

$$\psi(\tau) = \psi(0) + \int_0^\tau \lambda(\tau_1)\,d\tau_1 \tag{4.3-22}$$

This can be integrated using tabulated forms to the result

$$\begin{aligned} \psi(\tau) = -36.68^\circ + \sqrt{2}\Bigg\{&\frac{3}{\sqrt{2}}\lambda(\tau) - \tan^{-1}\frac{3}{\sqrt{2}}\lambda(\tau) \\ &- \left[\frac{3}{\sqrt{2}}\lambda(0) - \tan^{-1}\frac{3}{\sqrt{2}}\lambda(0)\right]\Bigg\} \end{aligned} \tag{4.3-23}$$

where $\lambda(\tau)$ is given by Eq. (4.3-21). The term $(3/\sqrt{2})\lambda(\tau)$ is in radian measure. Equations (4.3-15), (4.3-20), and (4.3-23) define the function $c(\tau)/V$, following the stated initial conditions. This function is plotted in Fig. 4.3-2, together with the exact solution. The agreement is seen to be excellent in spite of the substantial change in successive peaks in the early part of the response. The conditions of Eqs. (4.3-1) and (4.3-2), which should

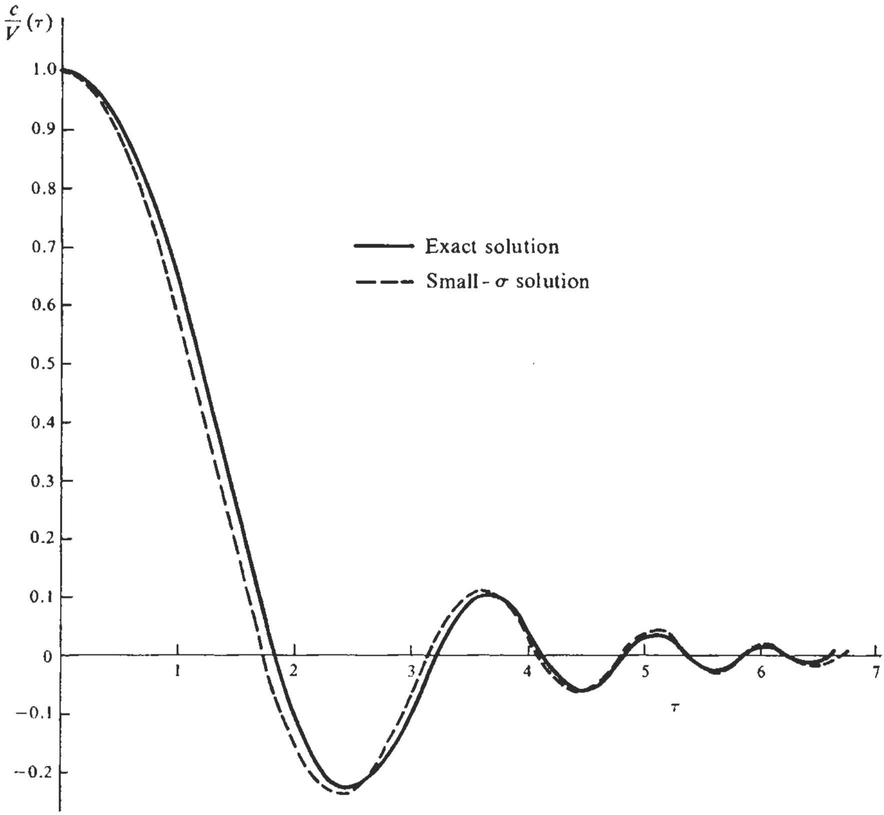

Figure 4.3-2 Solution of Example 4.3-1.

hold if one is to have confidence in characterizing the nonlinearity by its DF, are found to be badly violated in the early part of the response, when λ is small.

$$2\pi \frac{|\mu|}{\lambda}(0) = 4.67$$

$$2\pi \frac{|\dot{\lambda}|}{\lambda^2}(0) = 2.99$$

In spite of this, and the fact that the system linear part provides only second-order filtering of the harmonics in the nonlinearity output, the accuracy of the solution is excellent.

The quasi-static solution to this problem is readily found to be

$$\mu = -\tfrac{1}{2} \tag{4.3-24}$$

$$\lambda = \sqrt{\frac{4V}{\pi A} - \frac{1}{4}} \tag{4.3-25}$$

These values, obtained by solving Eqs. (4.3-7) after dropping the derivative terms, simply define the location of the closed-loop poles of this system in the p plane, considering the amplitude of the oscillation to be quasi-stationary. The damping factor μ in this solution is in error by 25 percent, which is just the contribution of $\dot{\lambda}$ in the equation of imaginaries.

In many cases of practical importance, the complete solution to $x(t)$ or $c(t)$ is not required; it is often sufficient just to know how the amplitude of the oscillation behaves. The solution for $A(t)$ is appreciably simpler than the solution for $x(t)$; we illustrate this situation in the following example.

Example 4.3-2 Consider the feedback system consisting of a relay-controlled third-order plant as shown in Fig. 4.3-3. We require the history of the amplitude of the limit cycle as it builds up from zero to its steady-state value. Application of the rules of Sec. 4.1 gives as the equations of reals and imaginaries

$$\text{Real:}\quad \frac{1}{\omega_n^2}(\sigma^3 - 3\sigma\omega^2 + 3\sigma\dot{\sigma} - 3\omega\dot{\omega} + \ddot{\sigma}) + \frac{2\zeta}{\omega_n}(\sigma^2 - \omega^2 + \dot{\sigma}) + \sigma + \frac{4KD}{\pi A} = 0 \tag{4.3-26a}$$

$$\text{Imaginary:}\quad \frac{1}{\omega_n^2}(3\sigma^2\omega - \omega^3 + 3\sigma\dot{\omega} + 3\omega\dot{\sigma} + \ddot{\omega}) + \frac{2\zeta}{\omega_n}(2\sigma\omega + \dot{\omega}) + \omega = 0 \tag{4.3-26b}$$

For the small-σ solution the equation of reals is written

$$\frac{1}{\omega_n^2}(-3\sigma\omega^2) + \frac{2\zeta}{\omega_n}(\sigma^2 - \omega^2) + \sigma + \frac{4KD}{\pi A} = 0 \tag{4.3-27}$$

which gives

$$\omega = \sqrt{\frac{4KD/\pi A + \sigma + (2\zeta/\omega_n)\sigma^2}{2\zeta/\omega_n + 3\sigma/\omega_n^2}} \tag{4.3-28}$$

Before taking the derivative of ω, we note that in writing the equation of imaginaries, it will be convenient to divide through by ω. The derivative of ω will then appear in the form

$$\frac{\dot{\omega}}{\omega} = \frac{d(\ln \omega)}{dt} = \frac{1}{2}\frac{-(4KD/\pi A)\sigma}{4KD/\pi A + \sigma + (2\zeta/\omega_n)\sigma^2} \tag{4.3-29}$$

ignoring the $\dot{\sigma}$ terms. Then dropping $\dot{\sigma}$ and $\ddot{\omega}$ according to the stated procedure for the small-σ solution, the equation of imaginaries is written

$$\frac{1}{\omega_n^2}\left[3\sigma^2 - \frac{4KD/\pi A + \sigma + (2\zeta/\omega_n)\sigma^2}{2\zeta/\omega_n + 3\sigma/\omega_n^2} - \frac{3}{2}\frac{(4KD/\pi A)\sigma^2}{4KD/\pi A + \sigma + (2\zeta/\omega_n)\sigma^2}\right] + \frac{2\zeta}{\omega_n}\left[2\sigma - \frac{1}{2}\frac{(4KD/\pi A)\sigma}{4KD/\pi A + \sigma + (2\zeta/\omega_n)\sigma^2}\right] + 1 = 0 \tag{4.3-30}$$

which is expanded to

$$\left(2 + 12\zeta^2 + 16\zeta^2\frac{1}{a}\right)\sigma^2 + 2\zeta\omega_n\left[1 + (1 + 6\zeta^2)\frac{1}{a}\right]\sigma + (2\zeta\omega_n)^2\left(1 - \frac{1}{a}\right)\frac{1}{a} = 0 \tag{4.3-31}$$

keeping only terms to σ^2. In this expression, the definition

$$a = \frac{\pi}{2}\frac{\zeta\omega_n}{KD}A \tag{4.3-32}$$

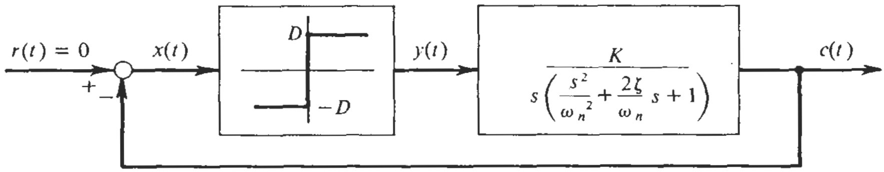

Figure 4.3-3 *System of Example 4.3-2.*

has been used. Note that, for this system, the steady-state limit cycle has $a = 1$, $\omega = \omega_n$, as can be derived from these equations by noting that $a = 1$ satisfies Eq. (4.3-31), with $\sigma = 0$, then evaluating ω from Eq. (4.3-28), with $a = 1$ and $\sigma = 0$.

Equation (4.3-31) is a nonlinear relation between a and $\sigma = \dot{a}/a$. A plot of $a\sigma$ versus a as defined by Eq. (4.3-31) is a phase-plane trajectory for the amplitude of the transient oscillation; this plot is shown in Fig. 4.3-4 for values of a ranging from 0.1 to 1.0. From this plot one can evaluate $a(t)$, using any of the standard methods of determining the time response from a phase-plane trajectory. (See, for example, Ref. 9, sec. 11.2.)

In the application to transient oscillations, the curves of $\dot{A} = A\sigma$ versus A for systems having a steady-state limit cycle generally have a near-linear characteristic for a substantial range of A around the limit cycle amplitude. In these cases it is convenient to fit the curve with one or more straight-line segments and treat each segment in the following manner. If

$$\dot{A} = m + nA \qquad (4.3\text{-}33)$$

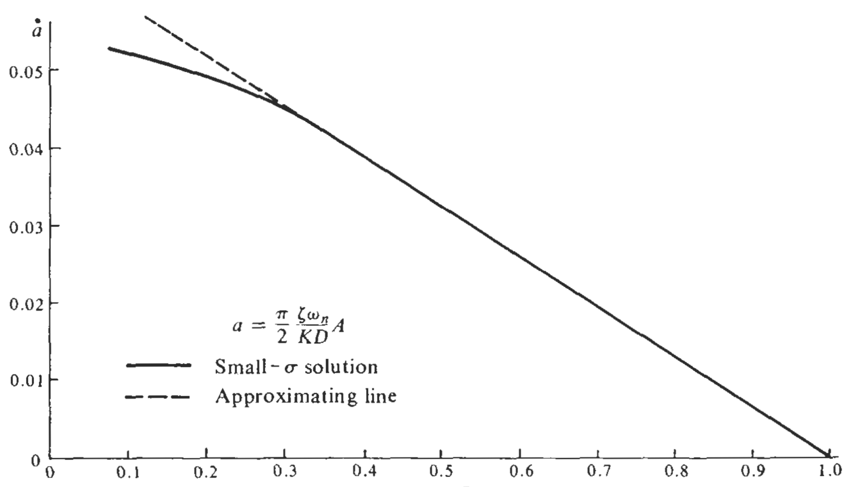

Figure 4.3-4 *Phase trajectory for the amplitude of the transient oscillation. Example 4.3-2, $\zeta = 5$, $\omega_n = 1$ rad/sec.*

over some interval of A, then the differential equation

$$\dot{A} - nA = m \tag{4.3-34}$$

having the solution

$$A(t) = -\frac{m}{n} + \left[\frac{m}{n} + A(0)\right]e^{nt} \tag{4.3-35}$$

holds over that interval. Exponential segments of $A(t)$ corresponding to linear segments of $\dot{A}(A)$ can then be pieced together, using a new time origin for each segment. Voronov (Ref. 10) has noted the usefulness of this procedure.

Example 4.3-2 (continued) For our present example, Eq. (4.3-31) as plotted in Fig. 4.3-4 can be fit quite well over the important part of its range by the single dashed line shown in the figure. The error in this fit at the small values of a is of little concern for two reasons: First, $\dot{a}$ is large in that region, so the value of a does not remain in that region long; and second, the original curve is not accurate at these small values of a because terms in σ^3, $\dot{\sigma}$, and $\dot{\omega}$ are not negligible there. Use of just the single approximating line gives

$$\dot{a} = 0.065 - 0.065a \tag{4.3-36}$$

and if this line is considered valid for the full interval of a ranging from 0 to 1, we have

$$a(t) = 1 - e^{-0.065t} \tag{4.3-37}$$

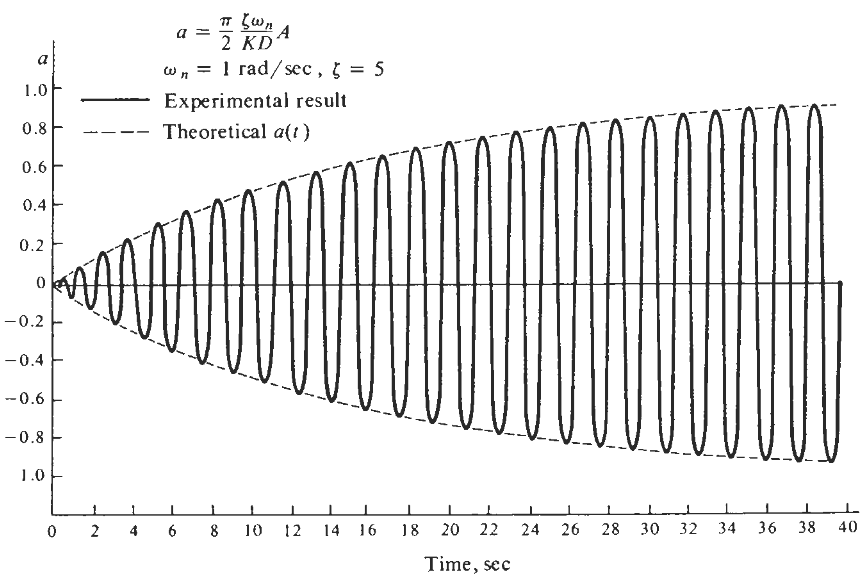

Figure 4.3-5 Solution of Example 4.3-2 (limit cycle build-up).

starting from the initial condition $a(0) = 0$. Actually, there is no analytic justification for presuming this solution to be valid for a less than 0.1; the solution effected here should be expected to define the limit cycle amplitude build-up over the range $a > 0.1$. However, it is found, when that solution is plotted with an experimentally observed build-up, that a continuation of the same exponential back to $a = 0$ continues to fit the observed data very well. The solution equation (4.3-37) is plotted in Fig. 4.3-5, together with a record of the limit cycle build-up in an analog simulation of this example system. The agreement is seen to be excellent.

4.4 LIMIT CYCLE DYNAMICS

A very important special case in the general study of transient oscillations is the subject of this section: the study of the dynamic characteristics of variations in limit cycles. This topic has particular relevance to the design of amplitude-regulating loops for limit cycling systems. The value of an automatic means of regulating the amplitude of a limit cycle in the presence of changing plant characteristics seems self-evident; the major considerations are the avoidance of large-amplitude limit cycles which would be costly, or uncomfortable, or perhaps dangerous, and the avoidance of loss of control capability, which usually accompanies very small amplitude limit cycles.

The general configuration of an amplitude-regulating loop is given in Fig. 4.4-1. The closed-loop limit cycling system is an operating control system. The amplitude of the limit cycle at some station in this system is sensed by an amplitude indicator, compared with the reference amplitude, and the error operated on to cause the adjustment of some parameter in the control system. The adjustment operator usually consists of an integrator plus any compensation that may be necessary to achieve the specified performance of the amplitude-regulating loop. The design of the regulating loop, including possible compensation, can be carried out in a direct manner only if the

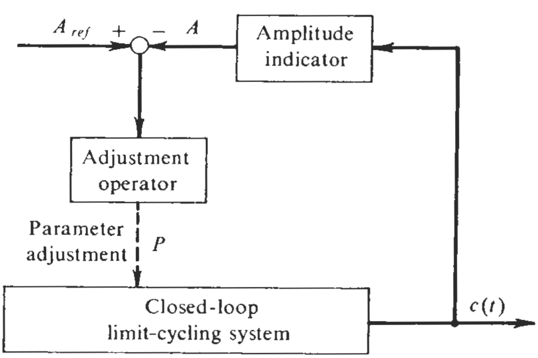

Figure 4.4-1 Limit cycle amplitude-regulating system.

dynamic response characteristics of each element in the loop are known and presented in a readily usable form. The adjustment operator is a standard linear device which presents no new problem. The amplitude indicator can be characterized to a good approximation by a readily derived transfer function (see, for example, Ref. 3, app. E). However, the dynamics of the change in limit cycle amplitude in response to a change in a parameter of the system is a more complicated matter. In this section we derive a transfer function to represent this dynamic characteristic.

The static sensitivity of the transfer from a parameter change, ΔP, to the resulting limit cycle amplitude change, ΔA, can be determined by the methods of Chap. 3. The steady-state limit cycle amplitude can be determined for several values of the parameter in the range of interest and the slope of the curve of A versus P taken as the static gain. But there is a dynamic effect as well, since the amplitude does not generally change instantaneously from one value to another when P is changed; there is usually a transient such as that pictured in Fig. 4.4-2. This transient is a transient oscillation, and can be analyzed by the methods of the preceding sections. However, it is of little value to the designer of an amplitude-regulating loop to know that he can plot the phase trajectory $\dot{A}$ versus A and thus deduce $A(t)$ for any particular situation. He needs a more readily usable description of the amplitude transient.

Such a description is indicated by the solution to the second example problem given in the preceding section. It was noted there that there was a substantial range of $\dot{A}$ around the limit cycle amplitude for which the curve of $\dot{A}$ versus A was very nearly linear, and that $A(t)$ was an exponential

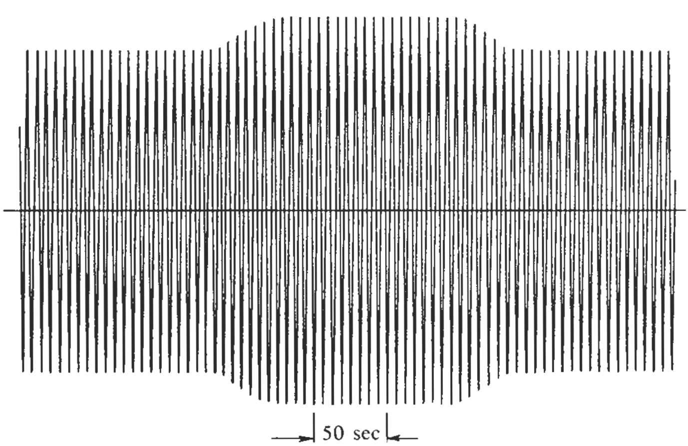

Figure 4.4-2 A limit cycle amplitude transient. System of Fig. 4.3-3, $\omega_n = 1$ rad/sec, $\zeta = 5$.

function over this range. Furthermore, in an amplitude-regulating loop which is operating continuously, one would not expect $\dot{A}$ to depart very far from zero; it would just take small positive and negative values as necessary to adjust the amplitude. It seems quite appropriate for this purpose, then, to approximate $\dot{A}$ as a linear function of A, taking the value zero at the limit cycle amplitude corresponding to the selected operating point and having the slope of the curve of $\dot{A}$ versus A at $\dot{A} = 0$.

$$\dot{A} = \left.\frac{d\dot{A}}{dA}\right|_{\dot{A}=0} (A - A_0) \tag{4.4-1}$$

A_0 is the steady-state limit cycle amplitude. This first-order differential equation is the characteristic equation of a first-order lag with time constant

$$\tau = -\left[\left.\frac{d\dot{A}}{dA}\right|_{\dot{A}=0}\right]^{-1} \tag{4.4-2}$$

The desired transfer function which characterizes the changes in limit cycle amplitude in response to changes in a system parameter P is then

$$\frac{\Delta A}{\Delta P} = \frac{(\Delta A/\Delta P)_{ss}}{\tau s + 1} \tag{4.4-3}$$

where the gain is the steady-state change in A for a given change in P, which can be evaluated by static methods as described before, and τ is given by Eq. (4.4-2).

A more convenient expression for the time constant is obtained by noting that

$$\begin{aligned}\left.\frac{d\dot{A}}{dA}\right|_{\dot{A}=0} &= \left.\frac{d(A\sigma)}{dA}\right|_{\sigma=0} \\ &= \left[A\frac{d\sigma}{dA} + \sigma\right]_{\sigma=0} \\ &= A_0 \left.\frac{d\sigma}{dA}\right|_{\sigma=0}\end{aligned}$$

Thus

$$\tau = -\left[A_0 \left.\frac{d\sigma}{dA}\right|_{\sigma=0}\right]^{-1} \tag{4.4-4}$$

INDIRECT SOLUTION FOR THE TIME CONSTANT

One way of determining this time constant is to follow the procedure for the small-σ solution given in Sec. 4.3 to the point in step 3 where one has an equation relating σ to A. For the second example problem this is Eq. (4.3-31). For the present purpose there is no need to plot the complete curve of $\dot{A}$ versus A; one can simply take the first derivative of each term in this equation and thus evaluate $d\sigma/dA$ at the point $\sigma = 0$, $A = A_0$. It is clear that terms in σ of higher degree than the first make no contribution to the result; so it is convenient to drop the higher-degree terms in σ at the outset, write the linear solution $\sigma(A)$, and take the first derivative.

For Eq. (4.3-31) this gives

$$\sigma = -\frac{2\zeta\omega_n(1 - 1/a)(1/a)}{1 + (1 + 6\zeta^2)(1/a)} \tag{4.4-5}$$

$$\left.\frac{d\sigma}{da}\right|_{a=1} = -\frac{2\zeta\omega_n}{2 + 6\zeta^2} \tag{4.4-6}$$

The time constant for the limit cycle amplitude transfer function for this system shown in Fig. 4.3-3 is

$$\tau = \frac{1 + 3\zeta^2}{\zeta\omega_n} \tag{4.4-7}$$

and when nondimensionalized with respect to the steady-state limit cycle period, $T_0 = 2\pi/\omega_n$, this becomes

$$\frac{\tau}{T_0} = \frac{1 + 3\zeta^2}{2\pi\zeta} \tag{4.4-8}$$

This relation has been checked experimentally by analog simulation for values of ζ ranging from 0.01 to 10. The limit cycle transient was excited by a step change in the forward gain of the system; the changing limit cycle was observed at the output of the linear part. The amplitude response in every case approximated an exponential very closely. Figure 4.4-2 shows this response for the case $\zeta = 5$. The time constant of the response was read as the time at which the amplitude transient completed 63 percent of its total excursion. These experimental results are plotted in Fig. 4.4-3, together with the theoretical curve of Eq. (4.4-8). The agreement is seen to be excellent over the full 1,000:1 range of damping ratio. It may be noted that the minimum time constant occurs for $\zeta = 1/\sqrt{3} = 0.577$, at which value $\tau/T_0 = 0.552$. In the mid-range of damping ratio, the amplitude transient is essentially completed within one period of the limit cycle, whereas systems with very high or very low damping ratios have amplitude transients lasting many periods.

The time constant for the amplitude transient based on the quasi-static solution of Eqs. (4.3-26) is easily derived (Ref. 4); the result is

$$\frac{\tau}{T_0} = \frac{1 + 4\zeta^2}{2\pi\zeta} \tag{4.4-9}$$

This expression is also plotted in Fig. 4.4-3. The distinction between the small-σ solution and the quasi-static solution in this case appears for the

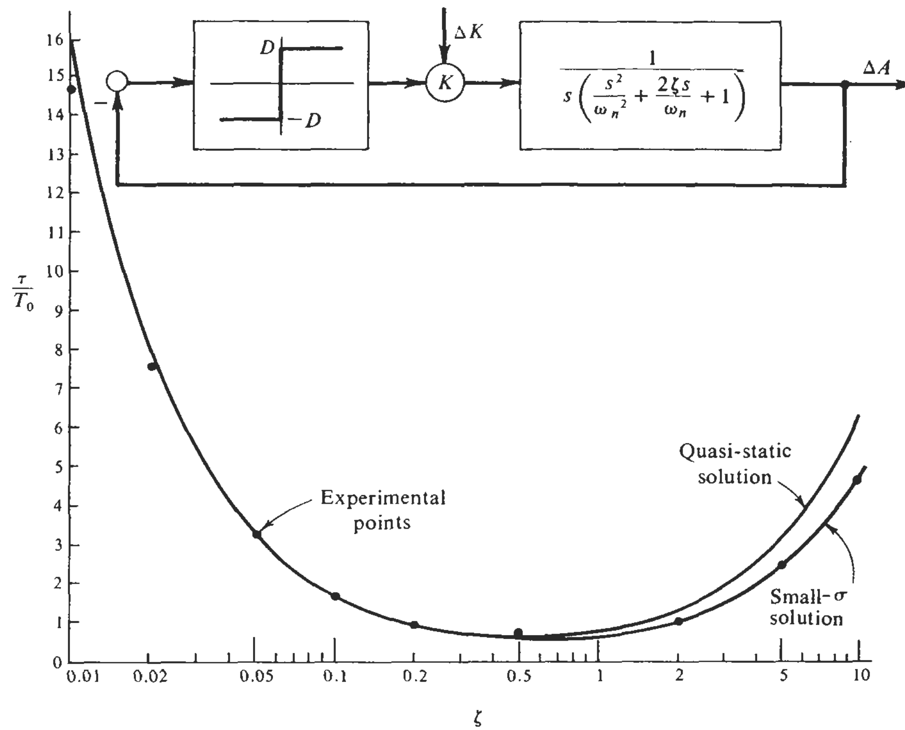

Figure 4.4-3 Time constant of limit cycle amplitude dynamics. System of Fig. 4.3-3.

large values of ζ only; the error in the quasi-static solution increases to 37 percent at $\zeta = 10$.

DIRECT SOLUTION FOR THE TIME CONSTANT

The solution given above for the time constant of the limit cycle amplitude transfer function was pursued primarily to emphasize the relationship of limit cycle dynamics to more general transient oscillations. For the determination of the amplitude transfer function time constant in practice, a more direct procedure is preferable. It is not necessary to solve the equation of reals for ω and put explicit expressions for ω and $\dot{\omega}$ into the equation of imaginaries; rather, one can write first differentials for the equations of reals and imaginaries directly and, with suitable assumptions, solve for $d\sigma/dA$, evaluated at the steady-state limit cycle condition.

If the equations of reals and imaginaries are written dropping derivatives of σ and derivatives of ω of higher order than one, they are of the form

$$R(\sigma,\omega,A,\dot{\omega}) = 0 \tag{4.4-10}$$

$$I(\sigma,\omega,A,\dot{\omega}) = 0 \tag{4.4-11}$$

Terms involving powers of σ greater than 1 can be dropped since they cannot influence the end result, which is the first derivative of σ evaluated at $\sigma = 0$. All powers of ω should be retained. In Eqs. (4.4-10) and (4.4-11), terms in σ, ω, and $\dot{\omega}$ enter directly, A and ω enter through n_p and n_q, and σ and $\dot{\omega}$ enter through $\dot{n}_p$ and $\dot{n}_q$. The relationship among first differentials is

$$\frac{\partial R}{\partial \sigma} d\sigma + \frac{\partial R}{\partial \omega} d\omega + \frac{\partial R}{\partial A} dA + \frac{\partial R}{\partial \dot{\omega}} d\dot{\omega} = 0 \tag{4.4-12}$$

$$\frac{\partial I}{\partial \sigma} d\sigma + \frac{\partial I}{\partial \omega} d\omega + \frac{\partial I}{\partial A} dA + \frac{\partial I}{\partial \dot{\omega}} d\dot{\omega} = 0 \tag{4.4-13}$$

According to the procedure for the small-σ solution, the equation of reals is solved for ω; in this case we solve for $d\omega$. It is no more difficult in this case to retain the effect of $d\dot{\omega}$.

$$d\omega = -\left(\frac{\partial R}{\partial \omega}\right)^{-1}\left(\frac{\partial R}{\partial \sigma} d\sigma + \frac{\partial R}{\partial A} dA + \frac{\partial R}{\partial \dot{\omega}} d\dot{\omega}\right) \tag{4.4-14}$$

From this we can write $\dot{\omega}$ as

$$\dot{\omega} = \frac{d\omega}{dt} = -\left(\frac{\partial R}{\partial \omega}\right)^{-1}\left(\frac{\partial R}{\partial \sigma}\frac{d\sigma}{dt} + \frac{\partial R}{\partial A}\frac{dA}{dt} + \frac{\partial R}{\partial \dot{\omega}}\frac{d\dot{\omega}}{dt}\right)$$

$$\approx -\left(\frac{\partial R}{\partial \omega}\right)^{-1}\left(\frac{\partial R}{\partial A} A\sigma\right) \tag{4.4-15}$$

since $\dot{\sigma}$ and $\ddot{\omega}$ terms are not being retained in the equations.

$$d\dot{\omega} = -\left(\frac{\partial R}{\partial \omega}\right)^{-1}\frac{\partial R}{\partial A} A_0\, d\sigma \tag{4.4-16}$$

at the steady-state limit cycle point, where $\sigma = 0$. Now these expressions for $d\omega$ and $d\dot{\omega}$ can be used in the equation of imaginaries [Eq. (4.4-13)]:

$$\frac{\partial I}{\partial \sigma} d\sigma + \frac{\partial I}{\partial \omega}\left(-\frac{\partial R}{\partial \omega}\right)^{-1}\left[\frac{\partial R}{\partial \sigma} d\sigma + \frac{\partial R}{\partial A} dA + \frac{\partial R}{\partial \dot{\omega}}\left(-\frac{\partial R}{\partial \omega}\right)^{-1}\frac{\partial R}{\partial A} A_0\, d\sigma\right]$$
$$+ \frac{\partial I}{\partial A} dA + \frac{\partial I}{\partial \dot{\omega}}\left(-\frac{\partial R}{\partial \omega}\right)^{-1}\frac{\partial R}{\partial A} A_0\, d\sigma = 0$$

This reduces to

$$\frac{d\sigma}{dA} = -\frac{\dfrac{\partial I}{\partial A} - \dfrac{\partial I}{\partial \omega}\left(\dfrac{\partial R}{\partial \omega}\right)^{-1}\dfrac{\partial R}{\partial A}}{\dfrac{\partial I}{\partial \sigma} - \dfrac{\partial I}{\partial \dot{\omega}}\left(\dfrac{\partial R}{\partial \omega}\right)^{-1}\dfrac{\partial R}{\partial A} A_0 - \dfrac{\partial I}{\partial \omega}\left(\dfrac{\partial R}{\partial \omega}\right)^{-1}\left[\dfrac{\partial R}{\partial \sigma} - \dfrac{\partial R}{\partial \dot{\omega}}\left(\dfrac{\partial R}{\partial \omega}\right)^{-1}\dfrac{\partial R}{\partial A} A_0\right]} \tag{4.4-17}$$

Using Eq. (4.4-4), we have, finally, as the time constant of the transfer function for amplitude variations,

$$\tau = \frac{\dfrac{1}{A_0}\dfrac{\partial I}{\partial \sigma} - \dfrac{\partial I}{\partial \dot{\omega}}\left(\dfrac{\partial R}{\partial \omega}\right)^{-1}\dfrac{\partial R}{\partial A} - \dfrac{\partial I}{\partial \omega}\left(\dfrac{\partial R}{\partial \omega}\right)^{-1}\left[\dfrac{1}{A_0}\dfrac{\partial R}{\partial \sigma} - \dfrac{\partial R}{\partial \dot{\omega}}\left(\dfrac{\partial R}{\partial \omega}\right)^{-1}\dfrac{\partial R}{\partial A}\right]}{\dfrac{\partial I}{\partial A} - \dfrac{\partial I}{\partial \omega}\left(\dfrac{\partial R}{\partial \omega}\right)^{-1}\dfrac{\partial R}{\partial A}} \tag{4.4-18}$$

This is an explicit expression for the time constant in terms of the partial derivatives of the real and imaginary functions evaluated at the steady-state limit cycle point: $\sigma = 0$, $\omega = \omega_0$, $A = A_0$, $\dot{\omega} = 0$.

The steady-state gain of the amplitude transfer function can be determined in a similar manner. The steady-state limit cycle is given by the solution of the equations of reals and imaginaries for the steady-state situation in which σ and all derivatives of σ and ω are zero. These equations are then just functions of A and ω, and of course the various parameters of the system. One of these parameters will be used to control the limit cycle amplitude; call that parameter P, and show explicitly the dependence of the real and imaginary functions on P. These functions, similar to those given in Eqs. (4.4-10) and (4.4-11), but including only the steady-state terms, are

$$R(A,\omega,P) = 0 \tag{4.4-19}$$

$$I(A,\omega,P) = 0 \tag{4.4-20}$$

We require the differential sensitivity relating changes in P to changes in A.

$$\frac{\partial R}{\partial A}\,dA + \frac{\partial R}{\partial \omega}\,d\omega + \frac{\partial R}{\partial P}\,dP = 0 \tag{4.4-21}$$

$$\frac{\partial I}{\partial A}\,dA + \frac{\partial I}{\partial \omega}\,d\omega + \frac{\partial I}{\partial P}\,dP = 0 \tag{4.4-22}$$

Elimination of the differential frequency change between these equations gives the desired result.

$$\frac{dA}{dP} = -\frac{\dfrac{\partial I}{\partial P} - \dfrac{\partial I}{\partial \omega}\left(\dfrac{\partial R}{\partial \omega}\right)^{-1}\dfrac{\partial R}{\partial P}}{\dfrac{\partial I}{\partial A} - \dfrac{\partial I}{\partial \omega}\left(\dfrac{\partial R}{\partial \omega}\right)^{-1}\dfrac{\partial R}{\partial A}} \tag{4.4-23}$$

This is the steady-state gain of the amplitude transfer function between the parameter P and the amplitude of the limit cycle at the input to the nonlinear part of the system. If the amplitude is to be monitored at some other point in the system, the transfer between that point and $x(t)$, to which the amplitude above applies, is a linear one. This yields a simple relation for the change in amplitude based on the change in amplitude at $x(t)$, and the change in frequency, which can also be determined from Eqs. (4.4-21) and (4.4-22).

Equations (4.4-23) and (4.4-18) give the values of the parameters for the amplitude transfer function of first-order form as shown in Eq. (4.4-3). The partial derivatives in Eq. (4.4-23) are derivatives of the steady-state functions [Eqs. (4.4-19) and (4.4-20)]. But these are identical with the corresponding derivatives of the more complicated functions [Eqs. (4.4-10) and (4.4-11)] when evaluated at the steady-state limit cycle point as required for use in Eq. (4.4-18). Thus, in practice, it is not necessary to write the steady-state equations and take derivatives; the same derivatives required in the evaluation of the time constant may be used to evaluate the gain.

These expressions for the time constant and static gain of the amplitude transfer function may seem somewhat formidable, but in practice the calculation and evaluation of the required partial derivatives is carried out more rapidly than alternative procedures requiring graphing of functions and the determination of slopes.

This more direct procedure for the determination of the amplitude transfer function is illustrated by application to an example system which is a bit more complicated than those considered previously. In this case, the nonlinearity has both real and imaginary parts; the limit cycle frequency as well as amplitude varies with the controlling parameter; the linear part of the system includes a lead term; and the controlling parameter is a time constant rather than the more common forward gain.

Example 4.4-1 This system is shown in Fig. 4.4-4. The variable used to adjust the limit cycle is the time constant of the lead term τ; thus the parameter P in Eq. (4.4-23) is in this case τ. The amplitude of the limit cycle is observed at the output $c(t)$, which is just the negative of $x(t)$, the input to the nonlinear part of the system. We proceed to derive the equations of the transient oscillation following the steps of Sec. 4.1. The system equation corresponding to Eq. (4.1-15) is

$$\left(\frac{1}{\omega_n^2}s^3 + \frac{2\zeta}{\omega_n}s^2 + s\right)X(s) = -(K + K\tau s)Y(s) \tag{4.4-24}$$

Application of the replacement rules gives

$$\frac{1}{\omega_n^2}(s^3 + 3s\dot{s} + \ddot{s}) + \frac{2\zeta}{\omega_n}(s^2 + \dot{s}) + s = -[KN + K\tau(Ns + \dot{N})] \tag{4.4-25}$$

which has real and imaginary parts

$$\text{Real:}\quad \frac{1}{\omega_n^2}(\sigma^3 - 3\sigma\omega^2 + 3\sigma\dot{\sigma} - 3\omega\dot{\omega} + \ddot{\sigma}) + \frac{2\zeta}{\omega_n}(\sigma^2 - \omega^2 + \dot{\sigma})$$
$$+ \sigma + Kn_p + K\tau(n_p\sigma - n_q\omega + \dot{n}_p) = 0 \tag{4.4-26a}$$

$$\text{Imaginary:}\quad \frac{1}{\omega_n^2}(3\sigma^2\omega - \omega^3 + 3\sigma\dot{\omega} + 3\omega\dot{\sigma} + \ddot{\omega}) + \frac{2\zeta}{\omega_n}(2\sigma\omega + \dot{\omega})$$
$$+ \omega + Kn_q + K\tau(n_p\omega + n_q\sigma + \dot{n}_q) = 0 \tag{4.4-26b}$$

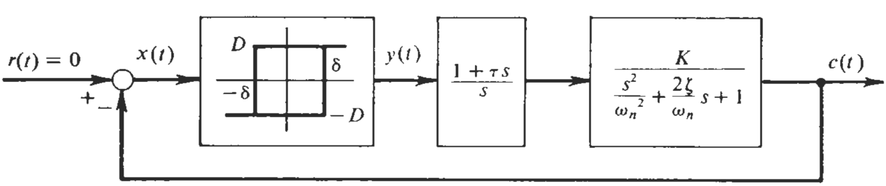

Figure 4.4-4 System of Example 4.4-1.

For the purpose of describing limit cycle dynamics, powers of σ greater than one are dropped, all derivatives of σ are dropped, and derivatives of ω of order greater than one are dropped.

$$R = \frac{1}{\omega_n^2}(-3\sigma\omega^2 - 3\omega\dot{\omega}) + \frac{2\zeta}{\omega_n}(-\omega^2) + \sigma + Kn_p + K\tau(n_p\sigma - n_q\omega + \dot{n}_p) \quad (4.4\text{-}27a)$$

$$I = \frac{1}{\omega_n^2}(-\omega^3 + 3\sigma\dot{\omega}) + \frac{2\zeta}{\omega_n}(2\sigma\omega + \dot{\omega}) + \omega + Kn_q + K\tau(n_p\omega + n_q\sigma + \dot{n}_q) \quad (4.4\text{-}27b)$$

Partial derivatives of these functions with respect to σ, ω, A, $\dot{\omega}$, and τ are to be written. The contributions of the DF for the nonlinear part of the system to these derivatives are seen from the explicit statements of functional dependence:

$$n_p = n_p(A,\omega) \quad (4.4\text{-}28)$$

$$n_q = n_q(A,\omega) \quad (4.4\text{-}29)$$

$$\dot{n}_p = \frac{\partial n_p}{\partial A} A\sigma + \frac{\partial n_p}{\partial \omega}\dot{\omega} \quad (4.4\text{-}30)$$

$$\dot{n}_q = \frac{\partial n_q}{\partial A} A\sigma + \frac{\partial n_q}{\partial \omega}\dot{\omega} \quad (4.4\text{-}31)$$

The derivatives are

$$\frac{\partial R}{\partial \sigma} = -\frac{3}{\omega_n^2}\omega^2 + 1 + K\tau n_p + K\tau\frac{\partial n_p}{\partial A}A$$

$$\frac{\partial R}{\partial \omega} = -\frac{6}{\omega_n^2}\sigma\omega - \frac{3}{\omega_n^2}\dot{\omega} - \frac{4\zeta}{\omega_n}\omega + K\frac{\partial n_p}{\partial \omega} + K\tau\sigma\frac{\partial n_p}{\partial \omega} - K\tau n_q - K\tau\omega\frac{\partial n_q}{\partial \omega} + K\tau\frac{\partial^2 n_p}{\partial A\,\partial \omega}A\sigma + K\tau\frac{\partial^2 n_p}{\partial \omega^2}\dot{\omega}$$

$$\frac{\partial R}{\partial A} = K\frac{\partial n_p}{\partial A} + K\tau\sigma\frac{\partial n_p}{\partial A} - K\tau\omega\frac{\partial n_q}{\partial A} + K\tau\frac{\partial n_p}{\partial A}\sigma + K\tau\frac{\partial^2 n_p}{\partial A^2}A\sigma + K\tau\frac{\partial^2 n_p}{\partial \omega\,\partial A}\dot{\omega}$$

$$\frac{\partial R}{\partial \dot{\omega}} = -\frac{3\omega}{\omega_n^2} + K\tau\frac{\partial n_p}{\partial \omega}$$

$$\frac{\partial R}{\partial \tau} = Kn_p\sigma - Kn_q\omega + K\dot{n}_p$$

$$\frac{\partial I}{\partial \sigma} = \frac{3}{\omega_n^2}\dot{\omega} + \frac{4\zeta}{\omega_n}\omega + K\tau n_q + K\tau\frac{\partial n_q}{\partial A}A$$

$$\frac{\partial I}{\partial \omega} = -\frac{3}{\omega_n^2}\omega^2 + \frac{4\zeta}{\omega_n}\sigma + 1 + K\frac{\partial n_q}{\partial \omega} + K\tau n_p + K\tau\omega\frac{\partial n_p}{\partial \omega} + K\tau\sigma\frac{\partial n_q}{\partial \omega} + K\tau\frac{\partial^2 n_q}{\partial A\,\partial \omega}A\sigma + K\tau\frac{\partial^2 n_q}{\partial \omega^2}\dot{\omega}$$

$$\frac{\partial I}{\partial A} = K\frac{\partial n_q}{\partial A} + K\tau\omega\frac{\partial n_p}{\partial A} + K\tau\sigma\frac{\partial n_q}{\partial A} + K\tau\frac{\partial n_q}{\partial A}\sigma + K\tau\frac{\partial^2 n_q}{\partial A^2}A\sigma + K\tau\frac{\partial^2 n_q}{\partial \omega\,\partial A}\dot{\omega}$$

$$\frac{\partial I}{\partial \dot{\omega}} = \frac{3}{\omega_n^2}\sigma + \frac{2\zeta}{\omega_n} + K\tau\frac{\partial n_q}{\partial \omega}$$

$$\frac{\partial I}{\partial \tau} = Kn_p\omega + Kn_q\sigma + K\dot{n}_q$$

These derivatives are to be evaluated at the steady-state limit cycle for the values of system parameters chosen as the operating point. The limit cycle is defined by the static equations

$$\frac{2\zeta}{\omega_n}(-\omega^2) + Kn_p - K\tau n_q \omega = 0 \tag{4.4-32a}$$

$$-\frac{1}{\omega_n^2}\omega^3 + \omega + Kn_q + K\tau n_p \omega = 0 \tag{4.4-32b}$$

The nonlinearity in this case is rectangular hysteresis, for which the DF was calculated in Chap. 2. Equation (2.3-26) gives

$$n_p = \frac{4D}{\pi A}\sqrt{1 - \left(\frac{\delta}{A}\right)^2} \tag{4.4-33a}$$

$$n_q = -\frac{4D\delta}{\pi A^2} \tag{4.4-33b}$$

The values selected for the nominal operating point are $\zeta = 2$, $\omega_n = 1$ radian/sec, $KD = 346$ volts/sec, $\delta = 10$ volts, $\tau = 0.22$ sec. A graphical solution of Eqs. (4.4-32) and (4.4-33) using these parameter values yields the results $A = 68.1$ volts, $\omega = 1.29$ radians/sec.

The steady-state limit cycle is then characterized by $A_0 = 68.1$ volts, $\omega_0 = 1.29$ radians/sec, $\sigma = \dot{\omega} = 0$. Using these values, the specified system parameters listed above, and Eqs. (4.4-33), the partial derivatives are found to be

$$\frac{\partial R}{\partial \sigma} = -3.96 \qquad \frac{\partial I}{\partial \sigma} = 10.53$$

$$\frac{\partial R}{\partial \omega} = -10.11 \qquad \frac{\partial I}{\partial \omega} = -2.58$$

$$\frac{\partial R}{\partial A} = -0.100 \qquad \frac{\partial I}{\partial A} = 0.0012$$

$$\frac{\partial R}{\partial \dot{\omega}} = -3.87 \qquad \frac{\partial I}{\partial \dot{\omega}} = 4.00$$

$$\frac{\partial R}{\partial \tau} = 1.228 \qquad \frac{\partial I}{\partial \tau} = 8.26$$

Use of these numbers in Eqs. (4.4-23) and (4.4-18) yields the theoretical transfer function relating changes in the lead-term time constant to changes in the amplitude of the limit cycle at the output of the system.

$$\frac{\Delta A}{\Delta \tau} = \frac{-298}{4.51s + 1} \tag{4.4-34}$$

An analog simulation of this system was tested by making step changes in the lead-term time constant between the values 0.20 and 0.24, which is a 20 percent change centered around the operating-point value used in the calculation above. As in the previous example, the time constant of the resulting amplitude response was taken to be the time in which the amplitude executed 63 percent of its change. The observed time constant was 4.4 sec, just 2.5 percent different from the predicted value. This small difference is within the measurement error for the experiment. The observed steady-state sensitivity based on the 20 percent change in τ was -263 volts/sec, which is 13 percent lower than the predicted value. The separate solution of the steady-state equations for $\tau = 0.20$ and 0.24 predicts the limit cycle amplitude in each case within 3 percent; so the 13 percent error above is attributable to the nonlinearity of the relation between τ and steady-state amplitude.

The description of limit cycle dynamics developed in this section has applied to the limit cycle as viewed at the input to the nonlinear part of the system, that is, at the point in the system which enjoys the full filtering effect of the linear part. If, however, the limit cycle is to be monitored at a point farther forward in the linear part, one would expect a somewhat different dynamic relation between the controlling parameter and the observed limit cycle amplitude. This may well be true, but experiment shows that as one observes at points farther forward, closer both to the nonlinearity and to the controlling parameter, the most dramatic change is that the whole nature of the amplitude response takes such a form as to defy description by a linear operator. The limit cycle amplitude response to a step change in forward gain may, for example, become quite sensitive to the phase angle in the limit cycle at which the step occurs, totally different responses resulting from a gain change at the zero crossing and at the peak of the cycle. From the point of view of the designer of an amplitude-regulating loop, this complication is of little consequence. The only change anticipated from the results derived here is the possibility of *less lag* in the amplitude dynamics for variables other than the input to the nonlinear part of the system. The results of this section then give a conservative estimate of the lag due to limit cycle dynamics, and may still be used to permit an analytical approach to the design of the system. That complicated phenomena such as those described above should occur in nonlinear systems is no surprise; the wonder is that the simple picture developed in this section does so well in characterizing the complex process of variations in the limit cycles of nonlinear systems.

We close this section with the observation that if the frequency, as well as the amplitude, of a limit cycle is to be monitored and perhaps controlled, the dynamics of limit cycle frequency changes are identical with the dynamics of amplitude changes, since the amplitude and frequency of a changing limit cycle are inextricably bound together as parts of a single transient oscillation.

4.5 LIMIT CYCLE STABILITY

We now return briefly to the matter of the stability of limit cycles, which was discussed earlier, in Sec. 3.2. At that point we had developed analytic tools for the description of steady-state oscillations only, and thus the stability of limit cycles was necessarily treated from a quasi-static point of view. That is to say, the truly dynamic properties of limit cycle transients which involve derivatives of σ and ω were not included in the description. It is clear that the "right" way to determine the stability of an indicated steady-state oscillation is to determine the stability of the *transient* oscillation which results when the steady-state oscillation is perturbed; to do so requires the analytic machinery of this chapter. However, in each of the examples of transient oscillations given in the preceding sections, the quasi-static description of the transient was found to be in error by 25 or 30 percent at most, and one should expect it to be a rare case in which the quasi-static description of a transient is so grossly in error as to give a false indication of stability. It is fortunately true that the techniques of Sec. 3.2 almost always give a correct indication of stability; one has to search to find examples in which they fail.

A more complete description of the stability of limit cycles has already been developed in the characterization of limit cycle dynamics given in the preceding section. There a transfer function of first-order form was derived which described the dynamic properties of the response of a limit cycle to a change in a parameter of the system. The time constant of this first-order transfer function was thought of as measuring the speed of response of the transient. As a by-product, it also indicates the stability of the limit cycle transient, and thus of the limit cycle. If the sign of τ given by Eq. (4.4-18) is positive, the limit cycle is stable; if τ is negative, the limit cycle is unstable. This is the familiar stability criterion for an ordinary first-order linear system; it applies equally well to the first-order linear approximation which was developed for limit cycle dynamics. From another point of view, note that a positive τ corresponds to a case in which the slope of $\dot{A}$ versus A is negative in the vicinity of the steady-state limit cycle where $\dot{A} = 0$. Thus if A is greater than A_0, $\dot{A}$ is negative and A decreases toward the steady-state value. Similarly, if A is less than A_0, $\dot{A}$ is positive and A increases toward A_0. On the other hand, if τ were negative, the slope of $\dot{A}$ versus A would be positive in the vicinity of $A = A_0$, $\dot{A} = 0$, and the same reasoning indicates a divergence of A away from A_0 following any disturbance.

It should be emphasized that although the sign of τ is a more complete indicator of limit cycle stability than any of the quasi-static methods, it still involves certain approximations. The original differential equations for A and ω derived in Sec. 4.1 incorporate the approximation of the nonlinear characteristic by its DF; in other respects these equations are exact. The small-σ solution of these equations is an approximate solution which ignores derivatives of σ and derivatives higher than the first of ω. This solution is further approximated in the expression for τ [Eq. (4.4-18)] by linearizing around the steady-state limit cycle point. The net difference between this stability criterion and any of the quasi-static methods given in Sec. 3.2 is then the inclusion of the effect of $\dot{\omega}$ on the dynamic properties of the system. The importance of this term to the *accuracy* of the description of transient oscillations was demonstrated in the examples given in the preceding sections. That the effect of $\dot{\omega}$ on the determination of the *stability* of a limit cycle can also be important is illustrated in the following example.

Example 4.5-1 Take as an example the system of Fig. 3.1-3. The open-loop linear part of this system is

$$L(s) = \frac{e^{-sT}}{s} \tag{4.5-1}$$

and the nonlinear part is

$$y = \delta x - \epsilon x^3 \tag{4.5-2}$$

or which the DF is

$$N(A) = \delta - \tfrac{3}{4}\epsilon A^2 \tag{4.5-3}$$

The form of these functions on a polar plot is shown in Fig. 4.5-1. The negative reciprocal DF for the nonlinear part of the system starts, for $A = 0$, at the point $-1/\delta$ and proceeds to the left for $A > 0$. It approaches minus infinity and changes sign at $A = \sqrt{4\delta/3\epsilon}$.

From Fig. 4.5-1 we observe a multiplicity of possible limit cycles, depending on the magnitude of δ (δ is shown as positive). The limit cycle at the point numbered 1 on the figure is possible if $-1/\delta$ lies inside the point where $L(j\omega)$ first crosses the negative real axis; that is,

$$\text{Limit cycle at 1 possible if } \delta > \frac{\pi}{2T} \tag{4.5-4}$$

Similarly,

$$\text{Limit cycle at 2 possible if } \delta > 2.5\,\frac{\pi}{T} \tag{4.5-5}$$

and so on for higher-frequency limit cycles.

The stability of these limit cycles as determined by any quasi-static method can readily be seen from the graphic approach. If the point $-1/[N(A)]$ is considered an equivalent -1 point for the purpose of the Nyquist stability criterion, we see that a limit cycle at the point numbered 1, if perturbed by increasing its amplitude, indicates a stable system; so

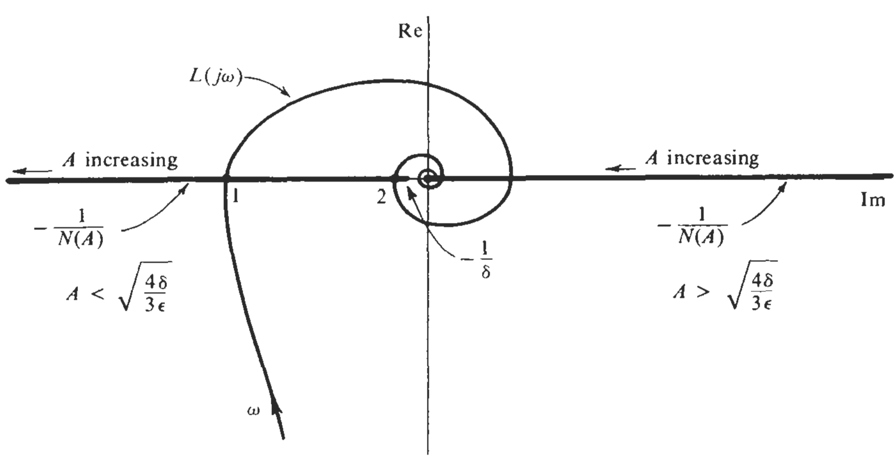

Figure 4.5-1 Polar plot of system of Example 4.5-1.

A would tend to decrease again. Similarly, if it is perturbed by decreasing its amplitude, it indicates an unstable system; so A would tend to increase again. These are the conditions for a stable limit cycle. Thus a quasi-static view of stability predicts that if δ is large enough to support the limit cycle mode numbered 1, that limit cycle will be stable no matter how large δ is. Experimental results show, however, that there is only a rather narrow band of values for δ for which this mode is possible and stable; this mode is actually unstable for all values of δ greater than some limit value—a behavioral pattern which a quasi-static description of stability fails to predict. The same argument shows the limit cycle mode numbered 2 and all higher-frequency limit cycles which may occur to be unstable.

We now test the stability of these limit cycles using the criterion of this section, the sign of the time constant of the transfer function for limit cycle dynamics. Defining $x(t)$, as usual, to be the input to the nonlinear part of the system, the differential equation which this closed-loop system obeys (see Fig. 3.1-3) is

$$\dot{x}(t + T) = -\delta x(t) + \epsilon x^3(t) \tag{4.5-6}$$

The substitution rules of Sec. 4.1 cannot be employed directly because of the time delay element in this system; we must address separately the matter of expressing $\dot{x}(t + T)$ as some function times $x(t)$, so that $x(t)$ can be divided out of each term of the equation as before. The form of the solution assumed is

$$x(t) = A(t)e^{j\psi(t)} \tag{4.5-7}$$

so

$$\dot{x}(t) = \dot{A}(t)e^{j\psi(t)} + j\omega(t)A(t)e^{j\psi(t)} \tag{4.5-8}$$

The purpose in using this form is that it expresses the solution in terms of $A(t)$ and $\omega(t)$, which we expect to be slowly varying functions. One may then expect that Taylor series expansions for $A(t + T)$ and $\omega(t + T)$ about the time t will be useful for the expression of $\dot{x}(t + T)$ in terms of variables evaluated at the time t. In fact, in the small-σ solution to be used, all derivatives of σ are ignored (corresponding to derivatives of A greater than the first), and derivatives of ω greater than the first are dropped. To be consistent with this, we shall use Taylor series expansions for $A(t + T)$ and $\omega(t + T)$ which retain only the first-derivative terms. Note that these expansions are used only for the purpose of expressing $\dot{x}(t + T)$ in a convenient form.

$$\dot{A}(t + T) \cong \dot{A}(t) \tag{4.5-9}$$

$$A(t + T) \cong A(t) + \dot{A}(t)T \tag{4.5-10}$$

$$\omega(t + T) \cong \omega(t) + \dot{\omega}(t)T \tag{4.5-11}$$

$$\psi(t + T) \cong \psi(t) + \omega(t)T + \tfrac{1}{2}\dot{\omega}(t)T^2 \tag{4.5-12}$$

Thus

$$\dot{x}(t + T) \cong [\dot{A} + j(\omega + \dot{\omega}T)(A + \dot{A}T)]e^{j(\psi + \omega T + \frac{1}{2}\dot{\omega}T^2)} \tag{4.5-13}$$

Each of the variables in the right-hand member is evaluated at the time t, and the functional dependence on t is omitted from the notation for the sake of brevity. Expanding and factoring out $Ae^{j\psi} = x(t)$ gives

$$\dot{x}(t + T) \cong [\sigma + j(\omega + \dot{\omega}T + \sigma\omega T + \sigma\dot{\omega}T^2)]x(t)e^{j(\omega T + \frac{1}{2}\dot{\omega}T^2)} \tag{4.5-14}$$

Using this expression and the DF for the x^3 nonlinear term $[N(A) = \tfrac{3}{4}A^2]$ in Eq. (4.5-6) and dividing $x(t)$ out of each term gives

$$[\sigma + j(\omega + \dot{\omega}T + \sigma\omega T + \sigma\dot{\omega}T^2)]e^{j(\omega T + \frac{1}{2}\dot{\omega}T^2)} + \delta - \tfrac{3}{4}\epsilon A^2 = 0 \tag{4.5-15}$$

which has real and imaginary parts

$$R = \sigma \cos(\omega T + \tfrac{1}{2}\dot{\omega}T^2) - (\omega + \dot{\omega}T + \sigma\omega T + \sigma\dot{\omega}T^2)\sin(\omega T + \tfrac{1}{2}\dot{\omega}T^2) + \delta - \tfrac{3}{4}\epsilon A^2 = 0 \tag{4.5-16a}$$

$$I = \sigma \sin(\omega T + \tfrac{1}{2}\dot{\omega}T^2) + (\omega + \dot{\omega}T + \sigma\omega T + \sigma\dot{\omega}T^2)\cos(\omega T + \tfrac{1}{2}\dot{\omega}T^2) = 0 \tag{4.5-16b}$$

The limit cycle modes are identified as the static or steady-state solutions of these equations. Using the static conditions $\sigma = \dot{\omega} = 0$, the solution is

$$\omega_0 = (m + \tfrac{1}{2})\frac{\pi}{T} \tag{4.5-17}$$

$$A_0 = \sqrt{\frac{4}{3\epsilon}\left[\delta - (-1)^m(m + \tfrac{1}{2})\frac{\pi}{T}\right]} \tag{4.5-18}$$

for m any nonnegative integer. The condition for a real oscillation is

$$\delta > (-1)^m(m + \tfrac{1}{2})\frac{\pi}{T} \tag{4.5-19}$$

All these steady-state results are of course identical with those derived earlier; the case $m = 0$ in Eq. (4.5-19) corresponds to Eq. (4.5-4), and $m = 2$ to Eq. (4.5-5).

The stability of these limit cycles is determined by the sign of the time constant of the limit cycle dynamics transfer function [Eq. (4.4-18)]. Toward the evaluation of this time constant, the required partial derivatives of the real and imaginary functions [Eqs. (4.5-16)] are calculated and evaluated at the limit cycle conditions $\sigma = \dot{\omega} = 0$, with ω given by Eq. (4.5-17) and A by Eq. (4.5-18). These derivatives are

$$\frac{\partial R}{\partial A} = -\tfrac{3}{2}\epsilon A_0 \qquad \frac{\partial I}{\partial A} = 0$$

$$\frac{\partial R}{\partial \sigma} = -(-1)^m(m + \tfrac{1}{2})\pi \qquad \frac{\partial I}{\partial \sigma} = (-1)^m$$

$$\frac{\partial R}{\partial \omega} = -(-1)^m \qquad \frac{\partial I}{\partial \omega} = -(-1)^m(m + \tfrac{1}{2})\pi$$

$$\frac{\partial R}{\partial \dot{\omega}} = -(-1)^m T \qquad \frac{\partial I}{\partial \dot{\omega}} = -(-1)^m\tfrac{1}{2}(m + \tfrac{1}{2})\pi T$$

which, when used in Eq. (4.4-18), determine the time constant to be

$$\tau = \frac{(-1)^m\{1/[(m + \tfrac{1}{2})\pi] + 2(m + \tfrac{1}{2})\pi\} - T\delta}{\tfrac{3}{2}\epsilon A_0^2} \tag{4.5-20}$$

This time constant is positive, and thus the limit cycle is indicated to be stable, if

$$\delta < \frac{(-1)^m}{T}\left[2(m + \tfrac{1}{2})\pi + \frac{1}{(m + \tfrac{1}{2})\pi}\right] \tag{4.5-21}$$

The condition for a real oscillation to exist was found to be

$$\delta > \frac{(-1)^m}{T}(m + \tfrac{1}{2})\pi \tag{4.5-22}$$

We are considering δ to be a positive quantity and are only interested in the positive frequency solutions; thus m is restricted to nonnegative even-integer values. For every such value of m there is a bounded range of δ for which a limit cycle is possible and stable, according to Eqs. (4.5-21) and (4.5-22):

For $m = 0$:
$$\frac{1.57}{T} < \delta < \frac{3.78}{T} \tag{4.5-23}$$

For $m = 2$:
$$\frac{7.85}{T} < \delta < \frac{15.83}{T} \tag{4.5-24}$$

For $m = 4$:
$$\frac{14.13}{T} < \delta < \frac{28.33}{T} \tag{4.5-25}$$

and so on for the higher-frequency limit cycles.

In this particular example, then, this dynamic stability criterion indicates an entirely different pattern of behavior for the system than do the quasi-static criteria. Computer experimentation with this system has demonstrated the behavior predicted above for the $m = 0$ limit cycle. The lower limit of δ for a limit cycle to exist is quite accurately predicted; the upper limit for the cycle to be stable was not determined accurately, but was found to lie between 4 and 5. The system is sensitive in that the initial amplitude must be taken rather close to the steady-state value if the system is to fall into the limit cycle mode. The higher-frequency limit cycle modes are not observed in the actual system. Multiple-input describing function theory shows clearly the reason for this. Although higher-frequency limit cycle modes are stable for limited ranges of δ, the system in the presence of these modes is unstable for any arbitrary small perturbation in addition to the limit cycle. Thus any additional perturbation diverges and prevents the continuation of the limit cycle. The analysis of this phenomenon requires the dual-input describing function of Chap. 6.

In this section we have presented a criterion for limit cycle stability which is more complete than those of Sec. 3.2 in that one more aspect of the dynamic character of a limit cycle transient, the effect of $\dot{\omega}$, is included. The fact that significantly new information is incorporated in this formulation is demonstrated by the example. It is not surprising that the more complete stability criterion is more laborious to apply than the simpler ones. Fortunately, the quasi-static indicators of stability are accurate in the vast majority of cases which present themselves in practice.

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PROBLEMS

4-1. Derive the replacement formulas, Eqs. (4.1-16) and (4.1-17), using the relation ($p = d/dt$)

$$p^n(xy) = \sum_{k=0}^{n} \binom{n}{k} (p^{n-k}x)(p^k y)$$

Start with

$$f_{n+1} = \frac{1}{x}(p^{n+1}x)$$

$$= \frac{1}{x} p^n(sx)$$

and

$$g_{n+1} = \frac{1}{x} p^{n+1}(Nx)$$

$$= \frac{1}{x}\{p^n[(pN)x + s(Nx)]\}$$

4-2. Suppose the solution for $x(t) = A(t) \exp[j\psi(t)]$ has been determined and we wish to observe $c(t)$. Using Eq. (4.1-23), with

$$L_1(s)L_3(s) = \frac{g_0}{d_0 + d_1 s}$$

write $c(t)$ in the form of a transient oscillation

$$c(t) = B(t) \exp[j\theta(t)]$$

and express $B(t)$ and $\theta(t)$ in terms of known functions. Note that the instantaneous amplitude and frequency of the transient oscillation at $c(t)$ are both different from those at $x(t)$.

4-3. Use the piecewise-sinusoidal-approximation technique described in Sec. 4.2 to calculate approximately the amplitude history for the limit cycle build-up of the system of Fig. 4.3-3. A root-locus plot is most convenient for determining the exponential damping factor corresponding to any amplitude of oscillation. Compare the result with Fig. 4.3-5.

4-4. The system of Fig. 4-1 starts from the initial condition $c(0)/KD = 100$, $\dot{c}(0) = 0$. Plot $A(t)/KD$ and $\omega(t)$, where $A(t)$ is the amplitude and $\omega(t)$ the frequency of the transient oscillation which $c(t)$ experiences.

Do both the small-σ and the quasi-static solutions. Discuss the range of applicability of these solutions. Can you improve either of them?

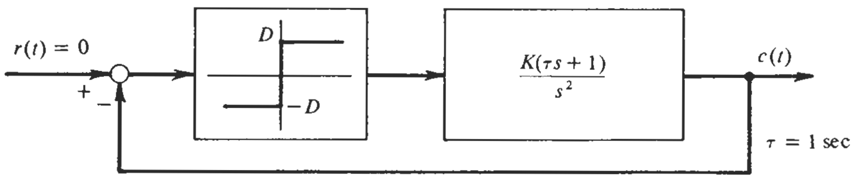

Figure 4-1

4-5. Find the small-σ solution to the differential equation

$$\frac{d^2y}{dt^2} + \alpha y^2 \frac{dy}{dt} + \omega_0^2 y = 0$$

You should be able to write out explicit expressions for $A(t)$ and $\omega(t)$ in terms of arbitrary initial conditions.

4-6. The system of Fig. 4-2 is turned on with an initial error of 50 units. Find the time history of the amplitude of the resulting transient oscillation as it decays from the initial 50 units to 5 units, after which the system operates in the linear region. What is the initial frequency of this oscillation and the frequency when the amplitude has decayed to 5 units? For simplicity in the solution for $\sigma(A)$, drop powers of σ greater than 1.

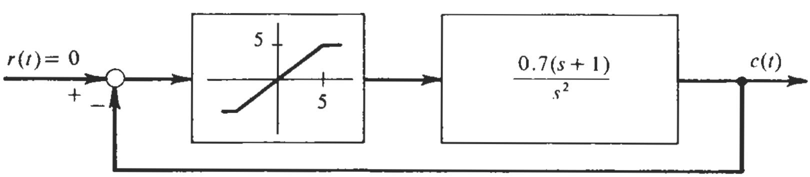

Figure 4-2

4-7. Nonspinning vehicles entering the earth's atmosphere at high speed exhibit oscillations in which the angle-of-attack history obeys approximately the following differential equation:

$$\ddot{\alpha} + M(\alpha)e^{at} = 0$$

$M(\alpha)$ is the nonlinear relation between angle of attack and restoring aerodynamic moment for a given dynamic pressure. The exponential term represents the increasing dynamic pressure as the vehicle descends into denser atmosphere.

$$M(\alpha) = 0.1\alpha - 0.02\alpha^3 \qquad a = 0.4 \text{ sec}^{-1}$$

Suppose the angle of attack at $t = 0$ is 1 radian. Assume the $\alpha(t)$ history to be in the form of a transient oscillation, and solve approximately for the amplitude and frequency of the oscillation. How long does it take for the amplitude to converge to 0.2 radian? What is the initial frequency and the frequency when the amplitude is 0.2 radian?

4-8. Derive the equations (the equations of reals and imaginaries) which the parameters of a transient oscillation in the system of Fig. 4-3 must obey. You may relate any delayed time function which appears in the problem to the present time function by assuming $\dot{A}$ and $\dot{\omega}$ are constant over the delay interval.

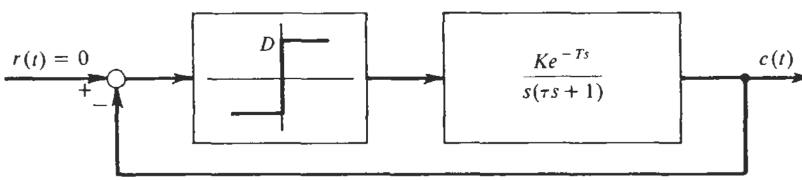

Figure 4-3

4-9. The system of Fig. 4-4 is in a steady-state limit cycle with $KD = 10$ units/sec. The forward gain is suddenly increased to $KD = 20$ units/sec. Plot the history of the amplitude and frequency of the transient in the limit cycle at $c(t)$.

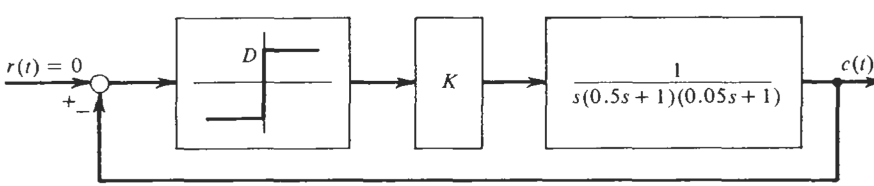

Figure 4-4

4-10. Use the direct-solution procedure [Eq. (4.4-18)] to find the time constant for limit cycle amplitude transients for the system of Fig. 4.3-3.