Chapter 6

Nonlinear conservative systems

The goal of this chapter is to formally introduce you to some adavnced concepts in the theory of nonlinear systems, especially the study of nonlinear systems that depend on a parameter. This will naturally lead us to the notion of catastrophe and bifurcation.

6.1 Local properties

The simplest conservative system may be written as the second-order equation

$$\ddot{x} = f(x), \tag{6.1}$$

where x usually is the position of, say, a mass and f is the sum of forces acting on this mass. Studying the trajectories of this system is made especially easy by multiplying the equation (6.1) by $\dot{x}$ to get

$$\frac{1}{2}\frac{d}{dt}\dot{x}^2 = \frac{d}{dt}\int_0^x f(\sigma)\, d\sigma. \tag{6.2}$$

Using the canonical coordinates $x_1 = x$, $x_2 = \dot{x}$, and noting $V(x) = -\int_0^x f(\sigma)d\sigma$, we can integrate (6.2) to obtain the integral equation

$$\frac{x_2^2}{2} + V(x_1) = h, \tag{6.3}$$

where h is an arbitrary constant. In fact, in most cases the term $x_2^2/2$ may be intrepreted as a kinetic energy, whereas the term $V(x_1)$ is interpreted as a potential energy. h is then the total energy for the system and it stays constant throughout the motion. Therefore, any trajectory has to lie on an *equi-energy curve* defined by (6.3). The equilibria of conservative systems correspond to the points where $x_2 = 0$ and $f(x_1) = 0$. Thus, equilibria correspond to the stationary values of the potential energy $V(x)$. There are only three possibilities, which are for V to either be a minimum, a maximum or an inflection point with horizontal tangent. Thus, the singular points of a conservative system may be classified in terms of the extremal properties of the potential energy at the singular points. In order to construct the phase portrait for conservative systems, one may immediately notice that (6.3) implies the phase portrait needs to be symmetric with respect to the x_1-axis.

Assume that the potential energy of the system, $V(x)$ is given. Then, writing

$$\frac{x_2^2}{2} = h - V(x_1), \tag{6.4}$$

it is possible to construct the equi-energy curve corresponding to a fixed value of h, and therefore the phase portrait. For each type of singularity, it is possible to build the corresponding shape of the equi-energy curves in the vicinity of that equilibrium.

Starting with minima, we see that the integral curves built in the vicinity of a minimum of the potential energy are closed curves, centered around the stable equilibrium of the system as shown in Fig. 6.1. These curves each correspond to limit cycles for specific values of the energy h. To determine the shape of these curves, one can rely on expansions of f and V. Assume that f can be expanded about an equilibrium point with $(\bar{x}, 0)$. We then can write

$$f(x) = a_1(x - \bar{x}) + \frac{a_2}{2!}(x - \bar{x})^2 + \frac{a_3}{3!}(x - \bar{x})^3 + \cdots$$

$V(x)$ may therefore be expanded as

$$V(x) = h_0 - \frac{a_1}{2!}(x - \bar{x})^2 - \frac{a_2}{3!}(x - \bar{x})^3 + \frac{a_3}{4!}(x - \bar{x})^4 - \ldots \tag{6.5}$$

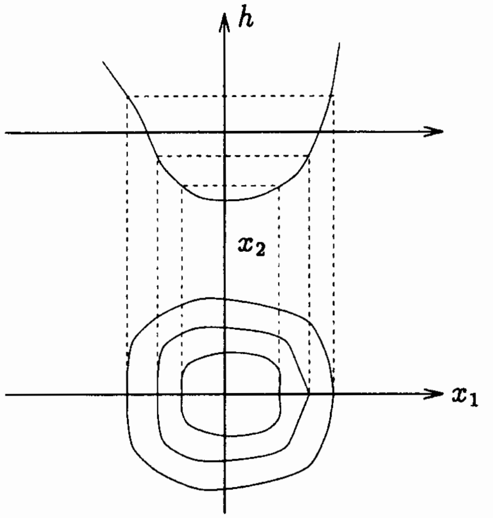

Figure 6.1: Stable equilibrium for a conservative system

such that locally around an equilibrium, the equi-energy curves need to satisfy the equation

$$\frac{x_2^2}{2} + h_0 - \frac{a_1}{2!}(x_1 - \bar{x})^2 - \frac{a_2}{3!}(x_1 - \bar{x})^3 + \frac{a_3}{4!}(x_1 - \bar{x})^4 - ... = h.$$

If the equilibrium is stable, then the first nonzero coefficient a_i must be *with odd index and negative.* If the first nonzero coefficient is a_1, then we see that an approximation of the equi-energy curves near equilibrium is

$$\frac{x_2^2}{2} + h_0 - \frac{a_1}{2!}(x_1 - \bar{x})^2 = h.$$

Whenever nonempty, these curves are *ellipsoids*, and the corresponding motion is an harmonic motion. The period of the motion is then independent from its amplitude, and the system locally looks like a linear mass-spring system. If a_1 is 0 and the equilibrium is stable, then we know $a_2 = 0$ and we look for a_3. a_3 nonzero and negative is illustrated by the mass-spring system shown in Fig. 4.10. In general, the equations characterizing the equi-energy curves for a stable equilibrium are of the

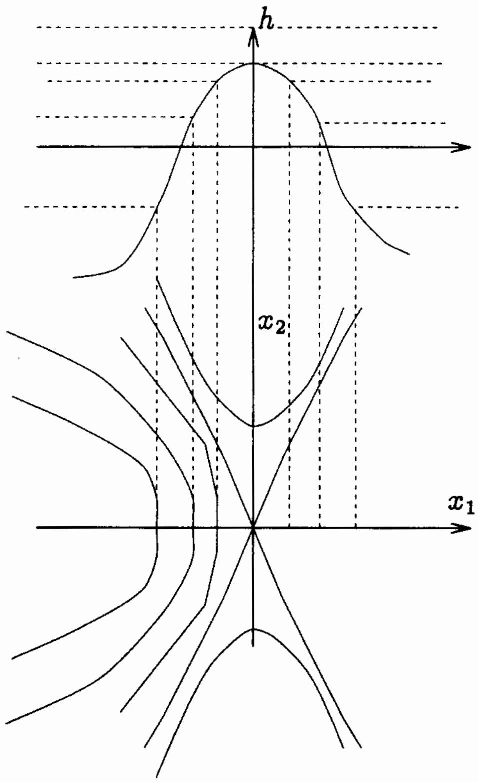

Figure 6.2: Unstable equilibrium for a conservative system

form

$$\frac{x_2^2}{2} + \kappa(x_1 - \bar{x})^{2p} = h - h_0,$$

and it is easily seen that, as p tends towards infinity, these curves look more and more like sausages elongated along the x_1-axis. Also, the period of these closed trajectories increases as their amplitude decreases.

When the equilibirum is *unstable*, then the equi-energy curves may again be computed using the equation (6.4), as shown in Fig 6.2. We now see that depending on the energy level h, the equi-energy curves in the vicinity of 0 do actually change shape. The separating curves are the ones that go through the origin. Using the expansion (6.5) of V around its maximum, we see that an unstable equilibrium can take place only if the first nonzero term of the expansion must be *with odd index and positive.* The general form of the equations for the equi-energy curves around the equilibrium is

$$\frac{x_2^2}{2} - \kappa(x_1 - \bar{x_1})^{2p} = h - h_0.$$

when $p = 1$, this corresponds to a family of hyperbolas whose focuses are either on the x_1 or x_2 axis. In general, the separatrices curves are then the two lines satisfying

$$\frac{x_2^2}{2} - \kappa(x_1 - \bar{x})^{2p} = 0,$$

which in the case $p = 1$ gives two lines with opposite slopes passing through the equilibrium point. When $p > 1$, this family is still made of hyperbola-like curves. However, the shape of the separatrices is now given by

$$x_2 = \pm 2\kappa(x_1 - \bar{x})^p,$$

and hence the two separatrices are tangent to the x_1-axis.

Equi-energy curves for a saddle point have been drawn in Fig. 6.3. Whenever the potential function $V(x)$ can be expanded about a saddle point, it is easy to see that its first nonzero coëfficient must be *with even order.* The separatrix then must satisfy an equation of the form

$$\frac{x_2^2}{2} - \kappa x_1^{2p+1} = 0,$$

or

$$x_2 = \pm\sqrt{\left|\kappa x_1^{2p+1}\right|}.$$

We can now state the two basic theorems relating stability of an equilibrium and the related potential energy V. We first have *Lagrange's theorem*, which says:

If in a state of equilibrium, the potential energy is minimum, then the equilibrium is stable.

We then have Lyapunov's converse theorem:

If in a state of equilibrium the potential energy is not a minimum, then the state of equilibrium is unstable

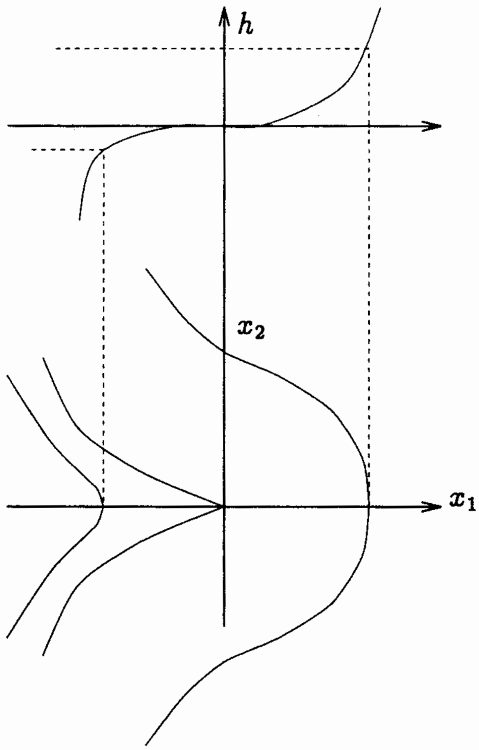

Figure 6.3: Equi-energy curves for a saddle point

6.2 Global properties

6.2.1 Separatrices

We now are interested in the global behavior of the conservative system

$$\ddot{y} = f(y),$$

once again by basing our analysis on the first integral of this equation, given by

$$\frac{\dot{y}^2}{2} + V(y) = h.$$

Depending on the relative positions of the curves $z = V(x)$ and $z = h$, several types of motion may occur for different values of the potential function as shown in Fig. 6.4. In particular, let us see what happens as we start from a high value of the total energy h and progressively let it go down. Here, we make the (not so bad) assumption that nothing really interesting happens outside the figure.

1. The line $z = h$ never intersects the curve $z = V(x)$. if the line $z = h$ constantly is below the line $z = V(x)$, then no motion can take place. If it lies above, then the resulting motion never has zero speed, and the two possible trajectories are symmetric with respect to the x_1-axis. These are the dashed surves in Fig. 6.4. As time tends to $+\infty$ or $-\infty$, such motions go to infinity as well, and they are named runaway motions. It is easily seen that the nature of such motions does not change with small perturbations of h.

2. The line $z = h$ tends towards the curve $z = V(x)$ as x tends to infinity. The resulting motion is once again infinite; however, as the position tends towards $+\infty$, speed goes down to 0. These are the fat dashed curves in Fig. 6.4. Note that this situation is *not* generic: if h moves up, then the motion changes (speed does not tend to 0), if h moves down, then the motion stops being infinite, as we see next.

3. The line $z = h$ intersects the curve $z = V(x)$ on one point: We now see that the corresponding trajectory is bounded on the right,

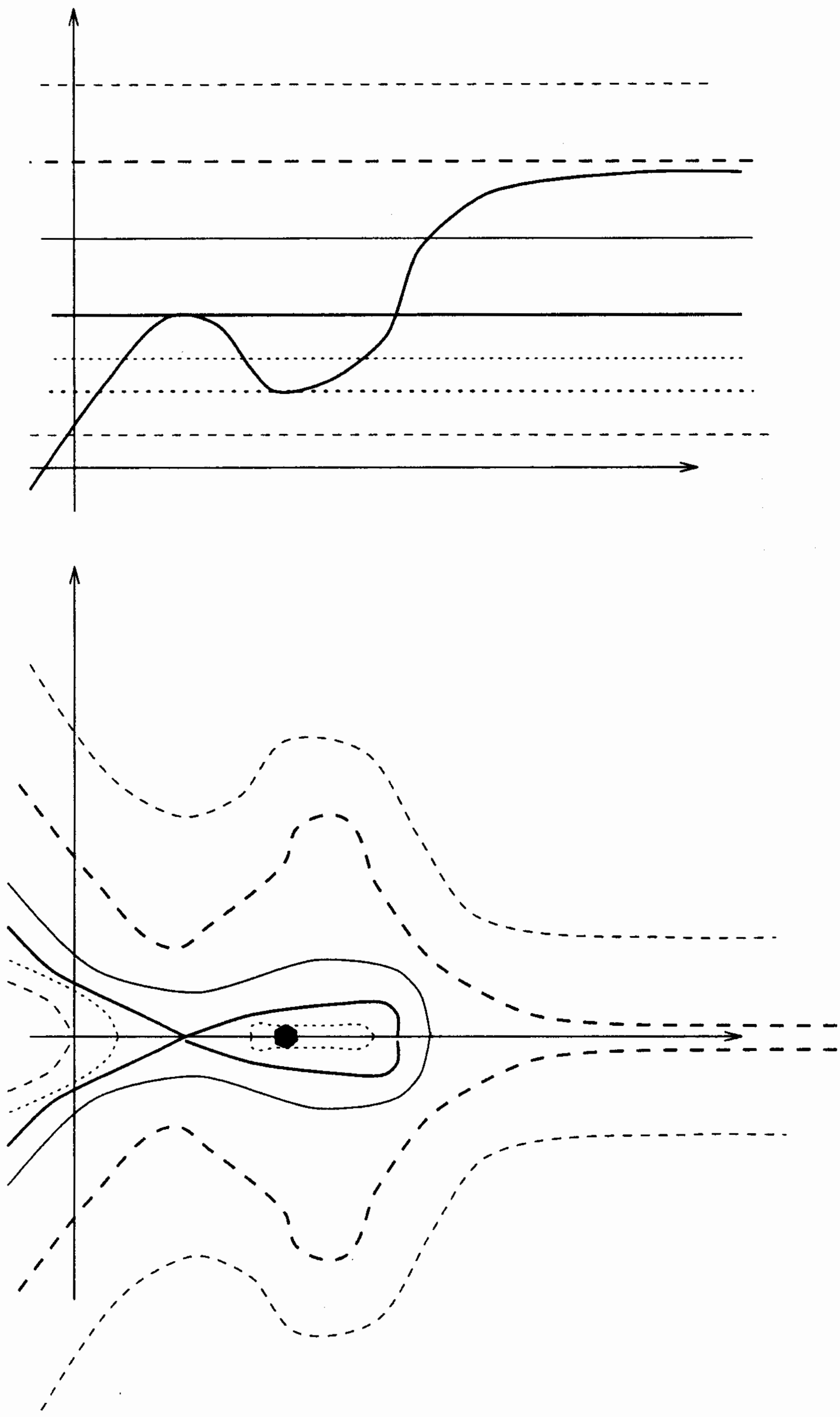

Figure 6.4: Building separatrices for conservative systems

while it is infinite on the left: states come from infinity on the left with positive speed and then return to infinity to the left, as shown by the continuous trajectories in Fig. 6.4. Note this situation is generic, that is the trajectories stay the same under small perurbations of h.

4. The line $z = h$ becomes tangent to one point of $V(z)$. Another set of trajectories now occurs: Previously unique trajectories now split into 4 distinct trajectories, shown in fat continuous curves. More precisely: one possible trajectory is the equilibrium point sitting on the local maximum of the potential function. The trajectory coming from $-\infty$ to the left with positive speed dies against that equilibrium. And the symmetric trajectory originates from the same equilibrium to go to $-\infty$. Finally, there is a closed trajectory initiating from the equilibrium and returning to it. Note that this situation is *non-generic*: it does not keep holding under small perturbations of h.

5. The line $z = h$ intersects the curve $z = V(x)$ in three points. The set of possible trajectories now reduces to two, as shown by the dotted curves in Fig. 6.4: we have one infinite trajectory to the left, and then one limit cycle. This situation is generic.

6. The line $z = h$ becomes tangent to the local minimum of the potential curve: there, once again only two trajectories may exist: There is an infinite trajectory to the left, and there also is a single equilibrium trajectory (a point). This situation is *nongeneric*.

7. The line $z = h$ cuts the curve $z = V(x)$ one one point and there is only one possible trajectory shown by the leftmost dashed curve.

This detailed analysis of the behavior of the conservative system shows that several phenomena occur: as h varies, the possible trajectories usually do not change nature, except at those special levels where $z = h$ becomes tangent to the potential function. Note these events are hard to observe, since they happen only at specific values of h. However, it suffices to know what the trajectories are at those points to be able to "interpolate" what all the other trajectories should be. These special trajectories form the "squeleton" of the whole phase-plane portrait. The trajectories emanating from the equilibria are named separatrices. Separatrices and equilibria are enough to build the rest of the phase portrait. They operate as "dividing" curves which separate regions with paths of different types.

6.2.2 Introduction to bifurcations: Dependence on a parameter

Let us now look at the previous problem from another angle: as h was varied from high to low values, it crossed certain very specific values (corresponding to equilibria and separatrices), where the aspect of the trajectories fundamentally changed nature: such *events* are generically named *catastrophes*, and they are directly related to the problem of computing the roots of $V(z) = h$ as h varies. Each time the number of real roots to that equation changes, the nature of the corresponding trajectories changes as well. Note the similarity of this problem with the root-locus problem. The problem is then to study when the roots of $1 + KG$ cross the $j\omega$-axis.

Let us see now how this notion of catastrophe may be extended to study *families* of conservative systems: in real-life applications, it is often that dynamical systems are uncertain, because some parameters that enter them are unknown. Thus it is important to know how the system should behave for all possible values of the parameter. In particular, we may expect to see *qualitative* changes in the phase portrait at some specific values of the parameter, while for most values of the same parameter, we will only encounter *quantitative* changes of the phase portrait. These specific values of the parameter will be named *bifurcation* or *branch* values of the parameter. We now show how to perform the bifurcation analysis of conservative systems. However, the bifurcation concept extends to nonconservative systems as well, at the expense of many more computations. For purposes of simplicity, we limit ourselves to studying the case of a dynamical system that depends on a single parameter only:

$$\ddot{x} = f(x, \lambda),$$

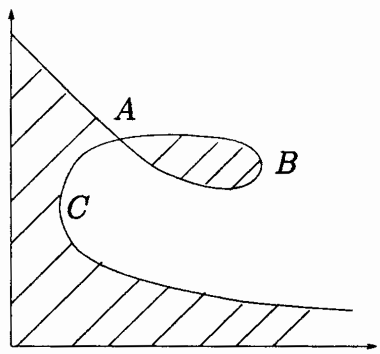

Figure 6.5: Generic bifurcation diagram

and we ask how the phase-plane portrait of the system changes as λ is varied. From the preceding developments, we know that the phase portrait of a conservative system is entirely determined by the position of the maxima and minima of V, that is, by the zeros of f. Thus, a qualitative change in the phase portrait should occur for those value of λ for which the pattern of the zeros of f changes. A plot that shows the position of the zeros of f vs. the value of the parameter λ is named a *bifurcation diagram.* Assume for example the bifurcation diagram shown in Fig. 6.5. Then to obtain the number of zeros of f, we just need to count the number of intersections of a vertical line $\lambda = \lambda_0$. Bifurcations occur when the number of roots for f changes. in this case, bifurcations occur at points A, B, and C.

In order to characterize analytically how the roots of f vary with the parameter λ, we can differentiate the equation

$$f(x, \lambda) = 0$$

to obtain

$$\frac{df}{dx}\frac{dx}{d\lambda} + \frac{df}{d\lambda} = 0.$$

Thus, we have

$$\frac{dx}{d\lambda} = -\frac{f'_\lambda(x, \lambda)}{f'_x(x, \lambda}.$$

So, if for a given value $\lambda = \lambda_0$, the system of equations $f(x, \lambda) = 0$, $f'_x(x, \lambda) = 0$ has no real solutions for x, then in a neighborhood of

$\lambda = \lambda_0$, the positions of all equilibria are continuous, differentiable functions of λ and λ_0 is not a bifurcation point. Assume now that at a given value of λ and x we have $f(x,\lambda) = 0$ and also $f'_x(x,\lambda) = 0$. If $f'_x(x,\lambda) = 0$ and $f'_\lambda(x,\lambda) \neq 0$, then the curve has at this point a vertical tangent. This corresponds to two values of zero curve merging and then becoming complex. If $f'_\lambda(x,\lambda)$ also vanishes, then we are in the situation shown at point A and we also have a bifurcation. Note the similarity with root-locus theory for linear systems!

In order to determine the stability of a point of equilibrium, we need to check the value of the second derivative of V, or, in other terms, the first derivative of f with respect to x. $f'_x(x,\lambda) > 0$ corresponds to a minimum of the potential energy, while $f'_x(x,\lambda) < 0$ corresponds to a maximum. Determining graphically the stability of equilibria may be done by shading those areas where $f(x,\lambda)$ is positive: every zero located under such a region will be unstable, whereas every zero located above such a region will be stable.

Note that the bifurcation diagram does provide some, but not all bifurcations for dynamical systems. Rather than going through an exhaustive list, we will present how the other bifurcations, which relate to the behavior of the separatrices occur in a practical example.

6.2.3 Example: Motion of a mass along a circle that rotates about a vertical axis (watt regulator)

We consider a mass m along a circle of radius a. The circle rotates about its vertical diameter with constant angular velocity Ω. The mass is subject to gravity, such that, noting θ the angular position of the mass, the equation of motion for the mass is

$$ma^2\frac{d^2\theta}{dt^2} = m\Omega^2 a^2 \sin\theta\cos\theta - mga\sin\theta.$$

Introducing the dimensionless parameter $\lambda = g/\Omega^2 a$ and performing the change of time $t_{\text{new}} = \Omega t$, we obtain the non-dimensionalized equation of motion

$$\ddot{\theta} + (\lambda - \cos\theta)\sin\theta = 0.$$

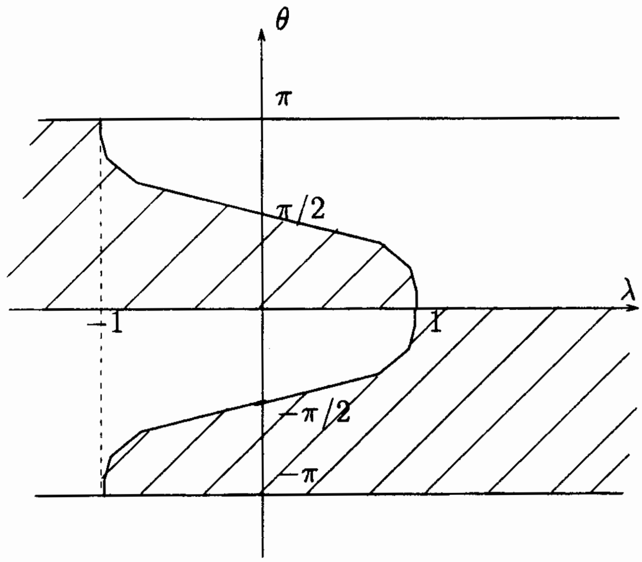

Figure 6.6: Bifurcation diagram.

Thus, the potential function is

$$V(\theta) = -\lambda \cos \theta - \frac{1}{2} \sin^2 \theta.$$

To study the bifurcations of this system, we plot the zeros of $(\cos(\theta) - \lambda) \sin \theta$ as a function of λ (these solutions are periodic with period 2π and we will limit ourselves to the interval $[-\pi \; \pi]$. It is obvious that the solutions $\theta = (-\pi, \; 0, \; \pi)$ always exist. Now for $|\lambda| \leq 1$, there also exist solutions $\theta = \pm \cos^{-1} \lambda$, where the function $\cos^{-1}$ maps the segment $[-1 \; 1]$ to the segment $[-\pi \; \pi]$. The resulting bifurcation diagram is shown in Fig. 6.6, and the shaded regions represent the regions where the derivative of V is negative. From this diagram, we may now immediately see that we have bifurcations for $\lambda = -1$ and $\lambda = 1$. Note that the variations of λ may either be interpreted as the variations of Ω or g. In particular, $\lambda \leq 0$ corresponds to either zero or negative gravity: the pendulum is upside down. When $\lambda > 1$, the system has two equlibria: a centre at $\theta = 0, \; \dot{\theta} = 0$, and a saddle point at $\theta = \pm\pi, \; \dot{\theta} = 0$. When $-1 < \theta < 1$, the system has four singular points: there are two centres at $\theta = \pm \cos^{-1} \lambda, \; \dot{\theta} = 0$ and two saddle points at $\theta = 0, \; \dot{\theta} = 0$ and $\theta = \pm\pi, \; \dot{\theta} = 0$. Finally, when $\lambda < -1$, we again have two equilibria only: a centre at $\theta = \pm\pi$, $\dot{\theta} = 0$ and a saddle point at $\theta = 0$, $\dot{\theta} = 0$.

To determine the separatrices' behavior, we will use the fact that each separatrix passes through a saddle point, for which the corresponding value of the total energy h may easily be evaluated. For $\lambda > 1$, there is one saddle point $\theta = \pm\pi$. The corresponding energy level is then $-\lambda\cos\pi - 0.5\sin^2\pi$, that is, λ. Therefore, the equation characterizing the separatrix is

$$\frac{1}{2}\dot{\theta}^2 - \frac{1}{2}(\sin^2\theta + 2\lambda\cos\theta) = \lambda,$$

or, in other terms,

$$\dot{\theta}^2 = \sin^2\theta + 2\lambda(1 + \cos\theta).$$

The phase portraits for all values of λ are given in Fig. 6.7. When $\lambda = 1$, the phase portrait actually does not change. Writing down the potential $V = \sin^2\theta + 2(1 + \cos\theta)$, and expanding it about 0, we obtain $V = 2/4!\theta^4 + \mathcal{O}(\theta^6)$; thus, we are in this situation where the system is stable around 0, yet the oscillations are not harmonic. You can easily check this is a case where the eigenvalues of the linearized model are both zero.

We see that, in addition to 1 and -1, $\lambda = 0$ is also a branch value for the phase portrait, corresponding to the change of relative sizes between unstable equilibria.

Eventhough it is hard to imagine, it suffices once again to know the phase portraits for the branch values of λ to be able to "interpolate" the phase portraits for the intermediate values of λ.

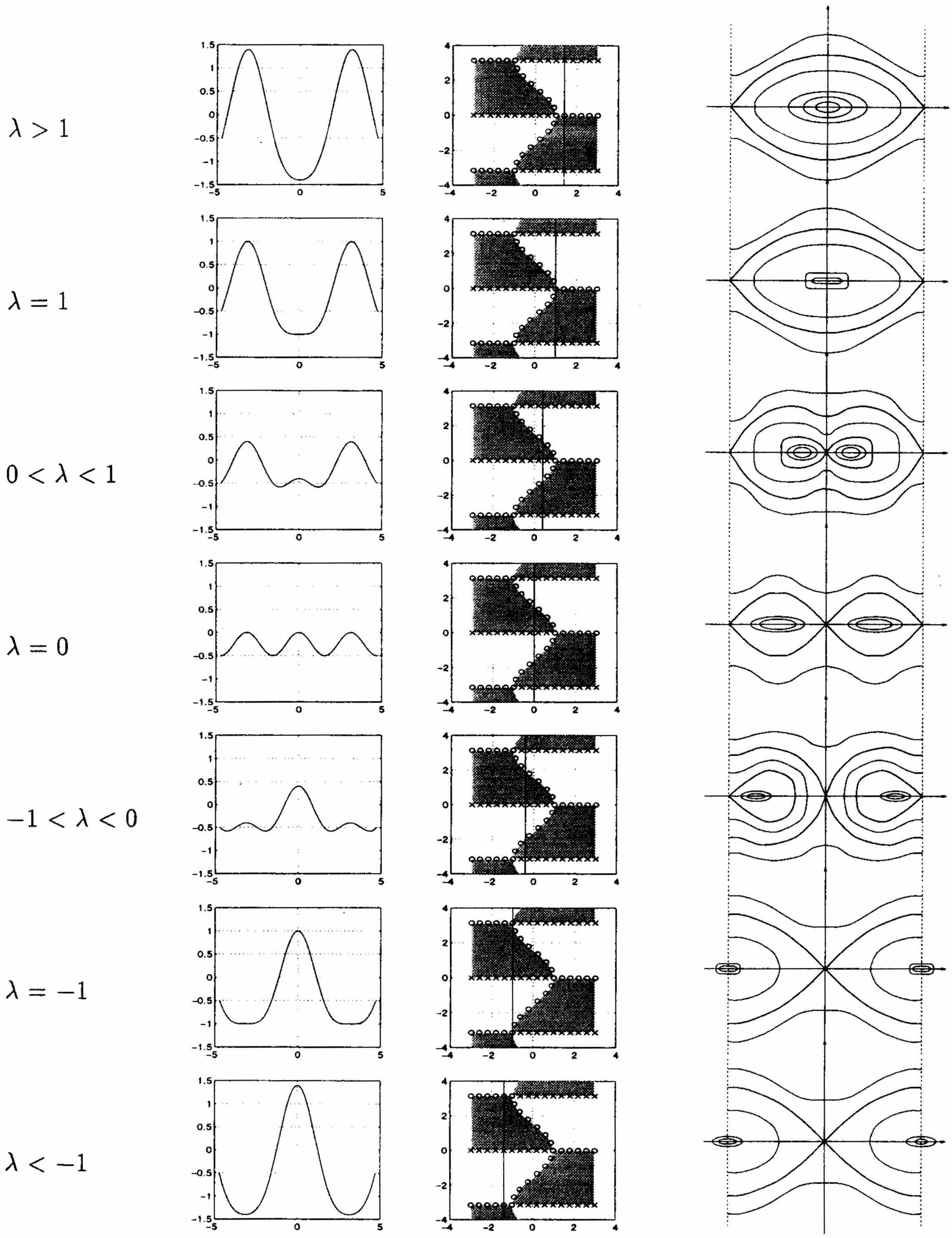

Figure 6.7: Phase portrait as λ changes.