Chapter 1

Introduction

Studying dynamical systems (whether they are electrical, mechanical, chemical or other) goes through the study of their *equations of motion*, often a set of ordinary differential equations. The goal of control engineering is, via a suitable input, to change these dynamics so that they behave in a satisfactory manner. When the dynamic system under study is *linear*, you have learned in detail a large number of approaches which are most likely to lead you to a good controller if it exists. Much of the success of these approaches is due to the fact that linear systems are "fairly simple"; in contrast, nonlinear systems, much more common in practice, may exhibit very complicated phenomena, which justify the existence of this course.

1.1 Linear systems

In your earlier control courses, you have (mostly) studied *linear, time-invariant* systems, or, in other terms, systems you can represent via transfer functions:

$$y(u) = H(s)u$$

Usually, you like to see H as a polynomial fraction: $H = n(s)/d(s)$, where n and d are polynomials in the complex variable s, and the degree of n does not exceed the degree of n. Let's try to remember what makes life easy with linear systems; well, linearity implies that the output produced by the sum of two inputs is the sum of the outputs

produced by each input separately: that is, $\forall (u_1, u_2)$,

$$
\begin{aligned}
y(u_1 + u_2) &= H(s)(u_1 + u_2) \\
&= H(s)u_1 + H(s)u_2 \\
&= y(u_1) + y(u_2).
\end{aligned}
$$

Also, the output produced by an input which has been scaled up (or down), is the scaled output: $\forall (u, \lambda)$,

$$
\begin{aligned}
y(\lambda u) &= H(s)(\lambda u) \\
&= \lambda H(s)u \\
&= \lambda y(u)
\end{aligned}
$$

This makes life extremely easy. In particular, once you know the behavior of the system for one signal u, you know the behavior of the system for λu, that is, the system is scale independent. Also, to check the stability of the system, you need to compute a few things only; namely, the roots of the characteristic polynomial $d(s)$. Once they have been determined to all have negative real part, then the system is stable. If one of them has positive real part, then the system is unstable.

By the way, do you remember what stability means? It can be given many senses, one of which is that bounded inputs should only generate bounded outputs. Consider for example the transfer function

$$
h(s) = \frac{10}{(s + 1)(s + 2)}
$$

and stick a square shaped input into it. The figure 1.1 clearly shows that the output does not grow big. And by linearity, we know this will still be true if the input is scaled up or down.

Another definition of stability comes from remembering that a linear system, though you've most often seen it as a transfer function, is before all an ordinary differential equation (ODE): for example, the transfer function

$$
h(s) = \frac{10}{(s + 1)(s + 2)}
$$

actually corresponds to the ODE
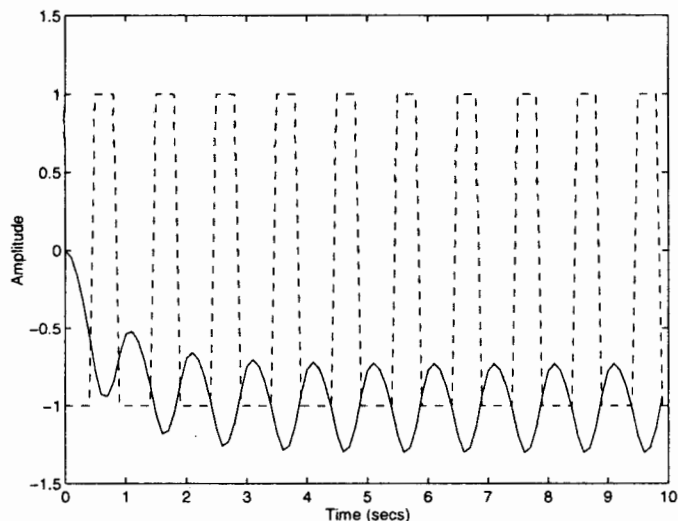
$$
\ddot{y} + 3\dot{y} + 2y = 10u. \tag{1.1}
$$

Figure 1.1: Illustration of bounded-input, bounded-output stability: Plain is output. Dashed is input.

Most probably, if this ODE came from a physically meaningful system, then y may have been the position (or voltage for an electrical system), $\dot{y}$ may have been the speed (intensity for the electrical system), and $\ddot{y}$ the acceleration. In which case we may as well rewrite (1.1) as a set of first-order differential equations by defining $x_1 = y$, $x_2 = \dot{y}$ and write

$$
\begin{aligned}
\frac{d}{dt}x_1 &= x_2 \\
\frac{d}{dt}x_2 &= -2x_1 - 3x_2 + 10u \\
y &= x_1
\end{aligned}
\tag{1.2}
$$

A characterization of linearity in this case is that the right hand sides of the equations above are all *linear* expressions in x_1 and x_2. We can now easily imagine a physical system that corresponds to (1.2), for example the mass-spring-dashpot system shown in Figure 1.2. Can you identify what the mass, the spring and the dashpot constants are? When looking at this physical system, u is now a force acting on the mass-spring system. This physical system suggests yet another definition of stability: assume that the force u is shut down to 0. The
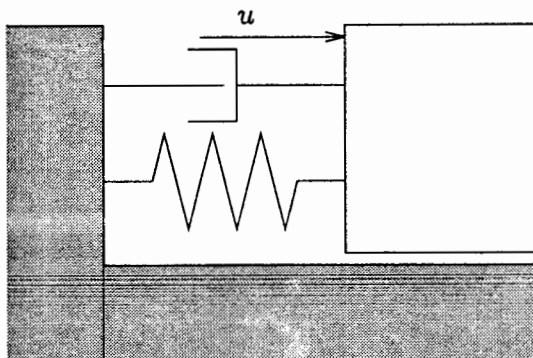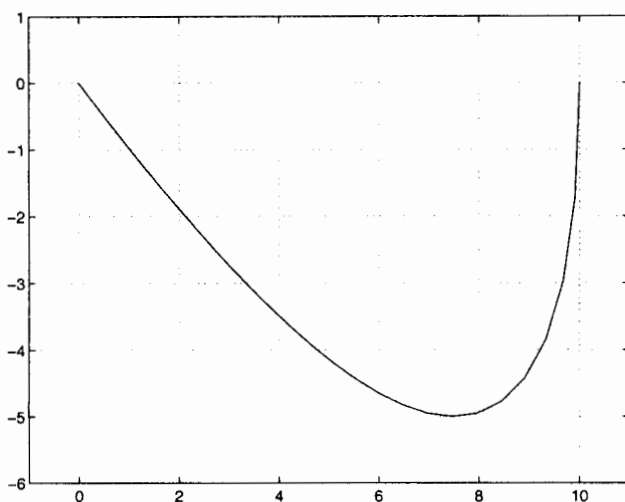
Figure 1.2: Mass-spring-dashpot system.

Figure 1.3: Stability: x-coordinate is position. y-coordinate is speed.

system has one quilibrium point, which is to stay at position 0 with
0 speed. If the mass is left anywhere (arbitrary initial position and
speed), it should eventually come back to 0 position with 0 speed. This
is illustrated in Figure 1.3, where we plotted x_1 and x_2, that is, the
position and speed of the mass in the system, starting from zero speed
and position equal to 10. It is interesting to note that this system, the
two notions of stability (bounded input, bounded output and stability
from given initial conditions) are completely *equivalent* for physically
meaningful systems. This idea generalizes to any physically meaningful

linear system

$$y = h(s)u, \tag{1.3}$$

where the notion of bounded-input, bounded-output stability is equivalent to the notion of stability from initial conditions, coming from the state-space representation of (1.3) as

$$\begin{aligned} \tfrac{d}{dt}x &= Ax + bu, \quad x(0) = x_0 \\ y &= cx + du \end{aligned}$$

1.2 Nonlinear systems

We now introduce the notion of nonlinear system by simply saying: it is any system which is not linear. One consequence is that nonlinear systems do not have an equivalent transfer function representation, and therefore we are bound to consider them in the form of nonlinear, ordinary differential equations (we will see later on in this course that in some particular situations it is possible to "cheat" and get a "frequency-response" for some nonlinear systems). You must understand that *most* systems you will encounter during your professional life are nonlinear; here are a few examples from various aero disciplines to convince yourself:

- **Mechanics: pendulum**

  The equations of motion for the pendulum are:

  $$L\ddot{\theta} + g\sin\theta = 0,$$

  where g is the gravity and L is the length of the pendulum. Of course, for small motions (θ near 0), you may want to linearize this equation,

  $$L\ddot{\theta} + g\theta = 0,$$

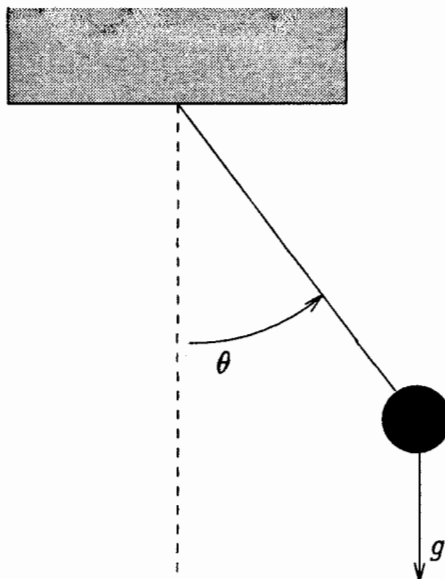 with reasonable confidence (and that's why linear control theory is so useful).
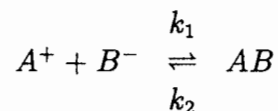
Figure 1.4: Pendulum.

- **Chemistry: Kinetics of reactions**

  Typical aqueous reactions involve the combination of 2 ions A^+ and B^- to form an insoluble combination AB according to the reaction:

  $$A^+ + B^- \underset{k_2}{\overset{k_1}{\rightleftharpoons}} AB$$

  The rates of the two opposite reactions k_1 and k_2 may be different, and the level of advancement of the reaction (*i.e.* the number of moles of produced or disappeared matter) C obeys the differential equation

  $$\frac{d}{dt}C = k_1 \left[A^+\right]\left[B^-\right] - k_2\left[AB\right].$$

  If the initial concentrations of each product were $\left[A_0^+\right]$, $\left[B_0^-\right]$ and $[AB_0]$, then the dynamics of C may be written

  $$\frac{d}{dt}C = k_1 \left(\left[A_0^+\right] - C\right)\left(\left[B_0^-\right] - C\right) - k_2\left([AB_0] + C\right),$$
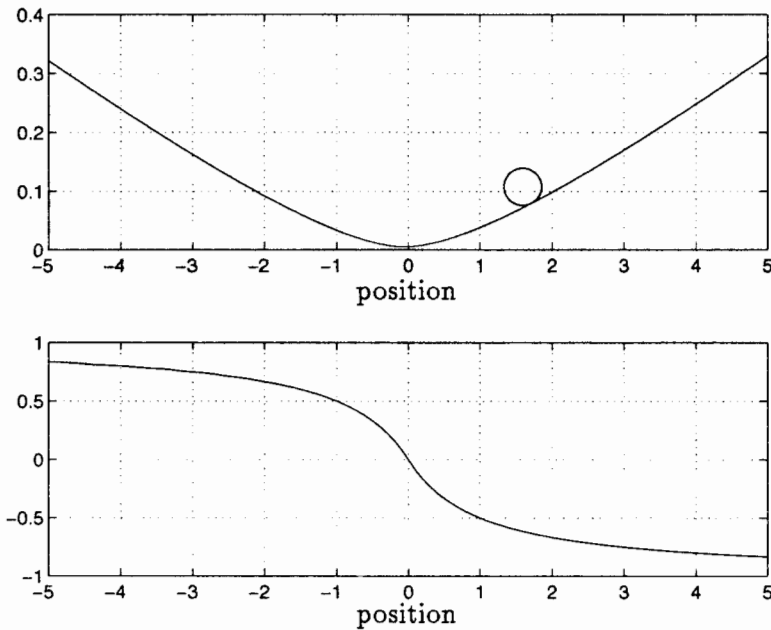
Figure 1.5: Top: the slope on which the cart slides; Bottom: horizontal acceleration along the slope.

which is clearly nonlinear (yet simple in this case).

Unlike linear systems, the notion of stability for nonlinear systems is much more complex: in particular, some nonlinear systems may be simultaneously seen as stable or unstable, and stability also depends on the definition attached to it! In addition, nonlinear systems behavior is often extremely sensitive to changes in initial conditions/inputs. We will now provide simple and physically meaningful illustrations of these facts by studying a cart sliding on a slope, subject to gravity and friction. Take a look at Figure 1.5; the cart (represented by a circle) is free to slide on the the line. Acting on this cart is gravity and viscous friction. The effect of gravity is given in the bottom figure; the equations of motion for the cart turn out to be represented by the differential equation

$$\ddot{y} + 0.1\dot{y} + \frac{y}{1 + |y|} = u, \tag{1.4}$$

where y is the horizontal position of the cart and u is a hypothetical external force. In the most common sense, this system is stable: it
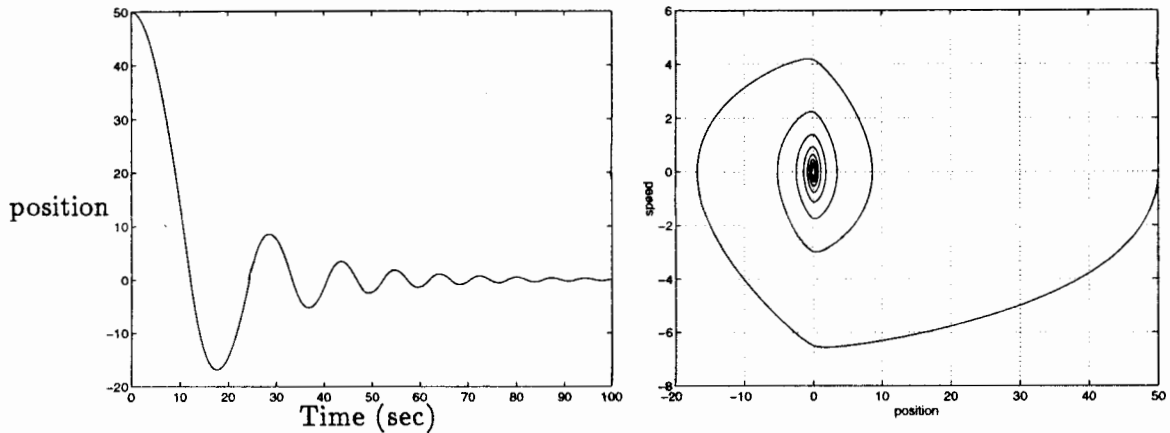
Figure 1.6: Cart position as a function of time and trajectory in the state-space.

does not necessarily show from the equation (1.4), yet it is obvious from physical insight: if you let it go from anywhere (in terms of speed and position), it will go back to 0, thanks to the restoring force. In Fig. 1.6, we have recorded the position of the cart as a function of time, starting with zero speed and position equal to 50. The result looks pretty much like a standard impulse response. Note however that the frequency of oscillations augments as the oscillations get smaller, an uncommon phenomenon when dealing with linear systems. Now, assume that a force acts on the cart. For a given maximum intensity, does the position of the cart stay finite? Well, possibly yes if the force intensity is less than gravity effects, but certainly not otherwise: it is obvious that the cart can climb anywhere if the push is big enough, yet finite. In other terms, stability for this system does not imply bounded-input, bounded-output stability, unlike linear systems.

Consider now the similar problem of a cart sliding on the slope shown in Fig. 1.7; The shape of the slope is a 3rd-order polynomial, such that the differential equation that drives the position of the cart in this case (assuming some viscous friction) is given by the following
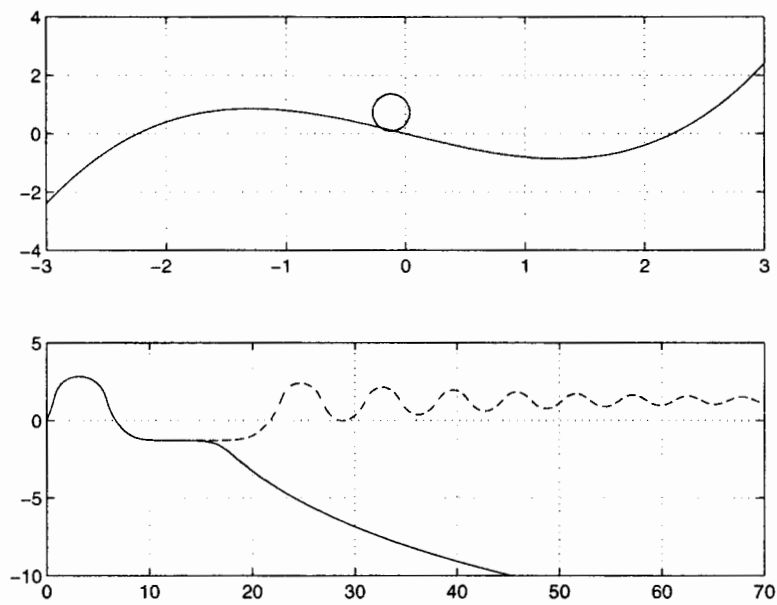
Figure 1.7: Top: the slope on which the cart slides; Bottom: two position histories from neighboring initial conditions; continuous: initial speed is 1.59139, dashed: initial speed is 1.59138.

set of nonlinear differential equations:

$$\frac{d}{dt}\frac{ds}{dt} = \frac{-0.6y^2 + 1}{\sqrt{1 + (0.6y^2 - 1)^2}} - 0.1\frac{ds}{dt} + u$$

$$\frac{d}{dt}y = \frac{ds}{dt}/\sqrt{1 + (0.6y^2 - 1)^2}. \tag{1.5}$$

In these equations, y is the horizontal position of the cart, and s is the corresponding curvilinear coordinate (they are related via the second equation in (1.5)). We now show another property of nonlinear differential systems: they are extremely sensitive to initial conditions or applied control: for example, the bottom plot shows the position of the free gliding cart as a function of time for two neighboring initial conditions; first, the cart is at coordinate zero and horizontal speed 1.59138. Second, the cart is at coordinate zero with horizontal speed 1.59139. As can be seen in Fig. 1.7, the resulting trajectories, though similar at the beginning, become rapidly completely different: the dashed curve shows some obvious stability properties, whereas the continuous curve does not look stable. In other terms, the system of equations (1.5) may exhibit stable or unstable behaviors. While looking at (1.5) does not give obvious information about this behavior, physical insight tells us what happens: if the cart is pushed at low enough speed on the right, it ends up reverting direction, but stays in the local "bowl" of the quadric. If the cart is pushed fast enough, it eventually gets off the ball to go away on the left.

1.3   Conclusions

In this chapter, we have seen a few of the differences between linear and nonlinear systems. We have seen through simple examples that nonlinear systems offer a vastly larger set of behaviors than linear systems. However, we also have seen that in many cases, physical insight helps tremendously in understanding these behaviors. This is nothing like linear systems, where you have learned tools that allow you to forget about the physics of the system and yet to design a **good** controller.

In the next lessons, we will elaborate on the ideas expressed in this chapter. In particular, we will spend a lot of time studying systematic

methods that will help you deal with nonlinear systems in your professional life. Remember, however, that nonlinear systems are not fully understood yet (those of you familiar with the Navier-Stokes equations of fluid dynamics know this).

Problems

1. Consider a pendulum with no friction. Describe *qualitatively* all the behaviors you may be able to observe for various initial conditions. Can you expect behaviours such as the one shown in Fig. 1.7?