16.21 Techniques of Structural Analysis and Design
Spring 2005
Unit #2 - Stress and Momentum balance

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Stress at a point

We are going to consider the forces exerted on a material. These can be external or internal. External forces come in two flavors: body forces (given per unit mass or volume) and surface forces (given per unit area). If we cut a body of material in equilibrium under a set of external forces along a plane, as shown in Fig.1, and consider one side of it, we draw two conclusions: 1) the equilibrium provided by the loads from the side taken out is provided by a set of forces that are distributed among the material particles adjacent to the cut plane and that should provide an equivalent set of forces to the ones loading the part taken out, 2) these forces can now be considered as external surface forces acting on the part of material under consideration.

The *stress vector* at a point on ΔS is defined as:

$$\mathbf{t} = \lim_{\Delta S \to 0} \frac{\mathbf{f}}{\Delta S} \tag{1}$$

If the cut had gone through the same point under consideration but along a plane with a different normal, the stress vector $\mathbf{t}$ would have been different. Let's consider the three stress vectors $\mathbf{t}^{(i)}$ acting on the planes normal to the coordinate axes. Let's also decompose each $\mathbf{t}^{(i)}$ in its three components in the coordinate system $\mathbf{e}_i$ (this can be done for any vector) as (see Fig.2):

$$\mathbf{t}^{(i)} = \sigma_{ij}\mathbf{e}_j \tag{2}$$

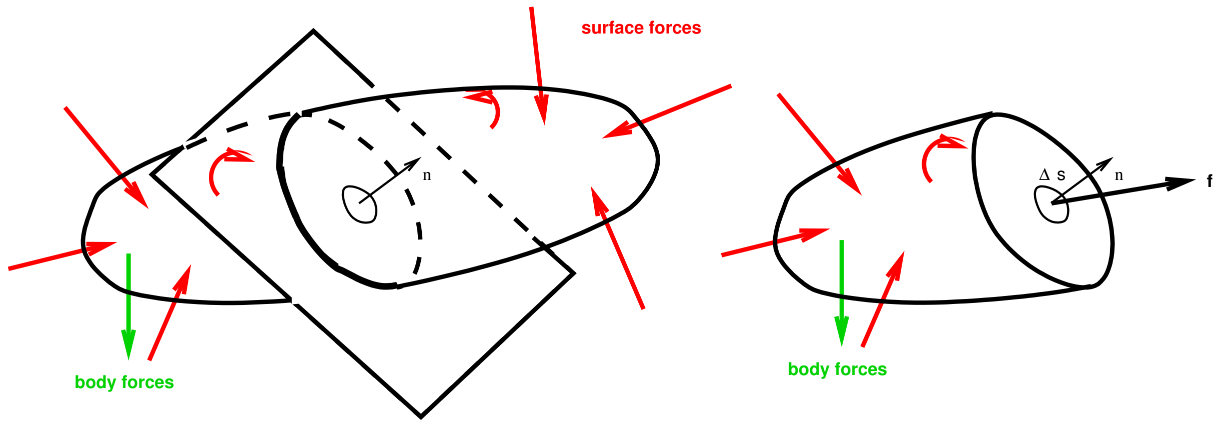

Figure 1: Surface force $\mathbf{f}$ on area ΔS of the cross section by plane whose normal is $\mathbf{n}$

σ_{ij} is the component of the stress vector $\mathbf{t}^{(i)}$ along the $\mathbf{e}_j$ direction.

Stress tensor

We could keep analyzing different planes passing through the point with different normals and, therefore, different stress vectors $\mathbf{t}^{(\mathbf{n})}$ and one might wonder if there is any relation among them or if they are all independent. The answer to this question is given by invoking equilibrium on the (shrinking) tetrahedron of material of Fig.3. The area of the faces of the tetrahedron are ΔS_1, ΔS_2, ΔS_3 and ΔS. The stress vectors on planes with reversed normals $\mathbf{t}(-\mathbf{e}_i)$ have been replaced with $-\mathbf{t}^{(i)}$ using Newton's third law of action and reaction (which is in fact derivable from equilibrium): $\mathbf{t}^{(-\mathbf{n})} = -\mathbf{t}^{(\mathbf{n})}$. Enforcing equilibrium we have:

$$\mathbf{t}^{(\mathbf{n})}\Delta S - \mathbf{t}^{(1)}\Delta S_1 - \mathbf{t}^{(2)}\Delta S_2 - \mathbf{t}^{(3)}\Delta S_3 = 0 \tag{3}$$

where ΔV is the volume of the tetrahedron and $\mathbf{f}$ is the body force per unit volume. The following relation: $\Delta S n_i = \Delta S_i$ derived in the following mathematical aside:

By virtue of Green's Theorem:

$$\int_V \nabla\phi dV = \int_S \mathbf{n}\phi dS$$

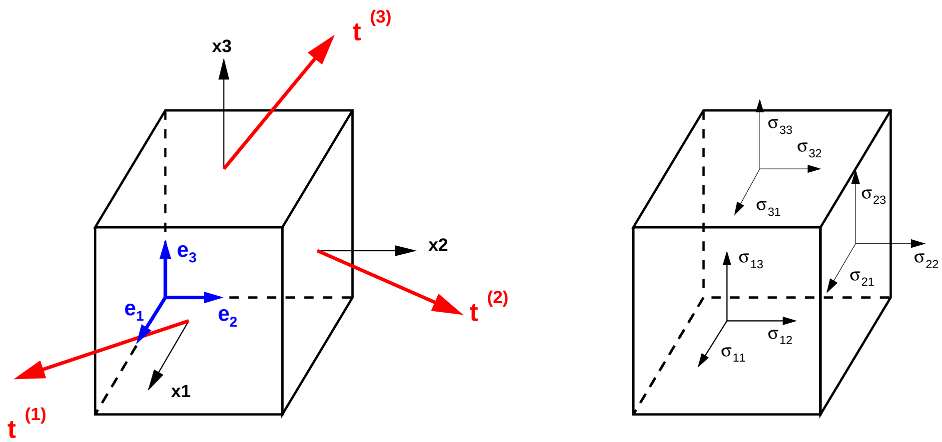

Figure 2: Stress components

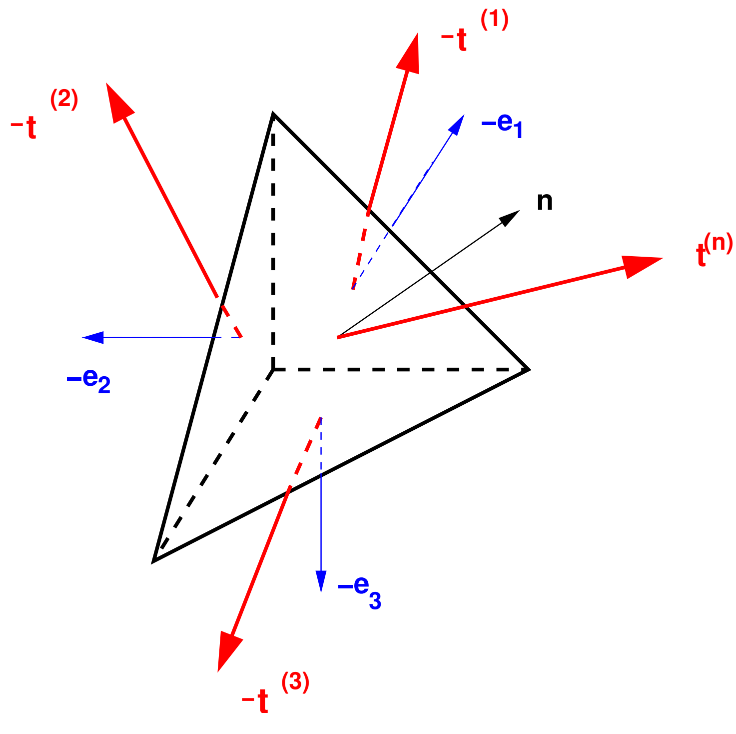

Figure 3: Cauchy's tetrahedron representing the equilibrium of a tetrahedron shrinking to a point

applied to the function $\phi = 1$, we get

$$0 = \int_S \mathbf{n} dS$$

which applied to our tetrahedron gives:

$$0 = \Delta S \mathbf{n} - \Delta S_1 \mathbf{e}_1 - \Delta S_2 \mathbf{e}_2 - \Delta S_3 \mathbf{e}_3$$

If we take the scalar product of this equation with $\mathbf{e}_i$, we obtain:

$$\Delta S (\mathbf{n} \cdot \mathbf{e}_i) = \Delta S_i$$

or

$$\boxed{\Delta S_i = \Delta S n_i}$$

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can then be replaced in equation 3 to obtain:

$$\Delta S \big(\mathbf{t}^{(\mathbf{n})} - (\mathbf{n} \cdot \mathbf{e}_1) \mathbf{t}^{(1)} + (\mathbf{n} \cdot \mathbf{e}_2) \mathbf{t}^{(2)} + (\mathbf{n} \cdot \mathbf{e}_3) \mathbf{t}^{(3)} \big) = 0$$

or

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \left(\mathbf{e}_1 \mathbf{t}^{(1)} + \mathbf{e}_2 \mathbf{t}^{(2)} + \mathbf{e}_3 \mathbf{t}^{(3)} \right) \tag{4}$$

The factor in parenthesis is the definition of the *Cauchy stress tensor* $\boldsymbol{\sigma}$:

$$\boxed{\boldsymbol{\sigma} = \mathbf{e}_1 \mathbf{t}^{(1)} + \mathbf{e}_2 \mathbf{t}^{(2)} + \mathbf{e}_3 \mathbf{t}^{(3)} = \mathbf{e}_i \mathbf{t}^{(i)}} \tag{5}$$

Note it is a tensorial expression (independent of the vector and tensor components in a particular coordinate system). To obtain the tensorial componenents in our rectangular system we replace the expressions of $\mathbf{t}^{(i)}$ from Eqn.2

$$\boldsymbol{\sigma} = \mathbf{e}_i \sigma_{ij} \mathbf{e}_j \tag{6}$$

Replacing in Eqn.4:

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\sigma} \tag{7}$$

or:

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \sigma_{ij} \mathbf{e}_i \mathbf{e}_j = \sigma_{ij} \big(\mathbf{n} \cdot \mathbf{e}_i \big) \mathbf{e}_j = \big(\sigma_{ij} n_i \big) \mathbf{e}_j \tag{8}$$

$$t_j = \sigma_{ij} n_i \tag{9}$$

Transformation of stress components

Consider a different system of cartesian coordinates $\mathbf{e}'_i$. We can express our tensor in either one:

$$\boldsymbol{\sigma} = \sigma_{kl}\mathbf{e}_k\mathbf{e}_l = \sigma'_{mn}\mathbf{e}'_m\mathbf{e}'_n \tag{10}$$

We would like to relate the stress components in the two systems. To this end, we take the scalar product of (10) with $\mathbf{e}'_i$ and $\mathbf{e}'_j$:

$$\mathbf{e}'_i \cdot \boldsymbol{\sigma} \cdot \mathbf{e}'_j = \sigma_{kl}\big(\mathbf{e}'_i \cdot \mathbf{e}_k\big)\big(\mathbf{e}_l \cdot \mathbf{e}'_j\big) = \sigma'_{mn}\big(\mathbf{e}'_i \cdot \mathbf{e}'_m\big)\big(\mathbf{e}'_n \cdot \mathbf{e}'_j\big) = \sigma'_{mn}\delta_{im}\delta_{nj} = \sigma'_{ij}$$

or

$$\boxed{\sigma'_{ij} = \sigma_{kl}\big(\mathbf{e}'_i \cdot \mathbf{e}_k\big)\big(\mathbf{e}_l \cdot \mathbf{e}'_j\big)} \tag{11}$$

The factors in parenthesis are the cosine directors of the angles between the original and primed coordinate axes.

Principal stresses and directions

Given the components of the stress tensor in a given coordinate system, the determination of the maximum normal and shear stresses is critical for the design of structures. The normal and shear stress components on a plane with normal $\mathbf{n}$ are given by:

$$\begin{aligned} t_N &= \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} \\ &= \sigma_{ki} n_k n_i \\ t_S &= \sqrt{\|\mathbf{t}^{(\mathbf{n})}\|^2 - t_N^2} \end{aligned}$$

It is obvious from these equations that the normal component achieves its maximum $t_N = \|\mathbf{t}^{(\mathbf{n})}\|$ when the shear components are zero. In this case:

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\sigma} = \lambda \mathbf{n} = \lambda \mathbf{I} \mathbf{n}$$

or in components:

$$\begin{aligned} \sigma_{ki} n_k &= \lambda n_i \\ \sigma_{ki} n_k &= \lambda \delta_{ki} n_k \\ \big(\sigma_{ki} - \lambda\delta_{ki}\big) n_k &= 0 \end{aligned} \tag{12}$$

which means that the principal stresses are obtained by solving the previous eigenvalue problem, the principal directions are the eigenvectors of the problem. The eigenvalues λ are obtained by noticing that the last identity can be satisfied for non-trivial $\mathbf{n}$ only if the factor is singular, i.e., if its determinant vanishes:

$$\begin{vmatrix} \sigma_{11}-\lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\lambda \end{vmatrix} = 0$$

which leads to the *characteristic equation*:

$$-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0$$

where:

$$I_1 = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} \tag{13}$$

$$I_2 = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}\right) = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right) \tag{14}$$

$$I_3 = det[\boldsymbol{\sigma}] = \|\sigma_{ij}\| \tag{15}$$

are called the *stress invariants* because they do not depend on the coordinate system of choice.

Linear and angular momentum balance

We are going to derive the equations of momentum balance in integral form, since this is the formulation that is more aligned with our "integral" approach in this course. We start from the definition of linear and angular momentum. For an element of material at position $\mathbf{x}$ of volume dV, density ρ, mass ρdV which remains constant, moving at a velocity $\mathbf{v}$, the linear momentum is $\rho\mathbf{v}dV$ and the angular momentum $\mathbf{x} \times (\rho\mathbf{v}dV)$. The total momenta of the body are obtained by integration over the volume as:

$$\int_V \rho\mathbf{v}dV \text{ and } \int_V \mathbf{x} \times \rho\mathbf{v}dV$$

respectively. The principle of conservation of linear momentum states that the rate of change of linear momentum is equal to the sum of all the external forces acting on the body:

$$\frac{D}{Dt}\int_V \rho\mathbf{v}dV = \int_V \mathbf{f}dV + \int_S \mathbf{t}dS \tag{16}$$

where $\frac{D}{Dt}$ is the total derivative. The lhs can be expanded as:

$$\frac{D}{Dt}\int_V \rho\mathbf{v}dV = \int_V \frac{D}{Dt}(\rho dV)\mathbf{v} + \int_V \rho\frac{\partial\mathbf{v}}{\partial t}dV$$

but $\frac{D}{Dt}(\rho dV) = 0$ from conservation of mass, so the principle reads:

$$\int_V \rho\frac{\partial\mathbf{v}}{\partial t}dV = \int_V \mathbf{f}dV + \int_S \mathbf{t}dS \tag{17}$$

Now, using what we've learned about the tractions and their relation to the stress tensor:

$$\boxed{\int_V \rho\frac{\partial\mathbf{v}}{\partial t}dV = \int_V \mathbf{f}dV + \int_S \mathbf{n}\cdot\boldsymbol{\sigma}dS} \tag{18}$$

This is the linear momentum balance equation in integral form. We can replace the surface integral with a volume integral with the aid of the divergence theorem:

$$\int_S \mathbf{n}\cdot\boldsymbol{\sigma}dS = \int_V \nabla\cdot\boldsymbol{\sigma}dV$$

and then (18) becomes:

$$\int_V \left(\rho\frac{\partial\mathbf{v}}{\partial t} - \mathbf{f} - \nabla\cdot\boldsymbol{\sigma}\right)dV = 0$$

Since this principle applies to an arbitrary volume of material, the integrand must vanish:

$$\rho\frac{\partial\mathbf{v}}{\partial t} - \mathbf{f} - \nabla\cdot\boldsymbol{\sigma} = 0 \tag{19}$$

This is the linear momentum balance equation in differential form. In components:

$$\sigma_{ji,j} + f_i = \rho\frac{\partial v_i}{\partial t}$$

Angular momentum balance and the symmetry of the stress tensor

The principle of conservation of angular momentum states that the rate of change of angular momentum is equal to the sum of the moment of all the external forces acting on the body:

$$\frac{D}{Dt}\int_V \rho\mathbf{x}\times\mathbf{v}dV = \int_V \mathbf{x}\times\mathbf{f}dV + \int_S \mathbf{x}\times\mathbf{t}dS \tag{20}$$

It can be conveniently written as

$$\int_S (x_i t_j - x_j t_i) dS + \int_V (x_i f_j - x_j f_i) dV = \int_V (x_i \frac{\partial v_j}{\partial t} - x_j \frac{\partial v_i}{\partial t}) dV$$

Using $t_i = \sigma_{ki} n_k$, the divergence theorem and (19), this expression leads to (see homework problem):

$$\int_V (\sigma_{ij} - \sigma_{ji}) dV = 0$$

which applies to an arbitrary volume V, and therefore, can only be satisfied if the integrand vanishes. This implies:

$$\boxed{\sigma_{ij} = \sigma_{ji}} \tag{21}$$