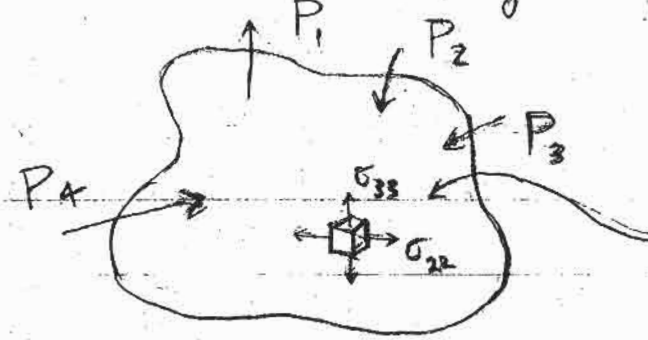
FAILURE, FRACTURE, AND FATIGUE

Basic Structures Problem

Given an arbitrary body —



Displacement $\bar{u} = u_n \bar{i}_n$

In Tensor Notation —

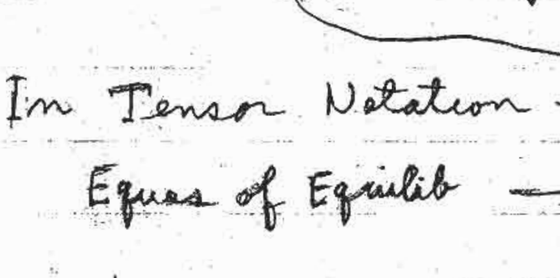
Equns of Equilib $\longrightarrow$ $\dfrac{\partial \sigma_{mn}}{\partial x_m} + f_n = 0$ (3 Eqs)

Strain-Displacement Eqs $\longrightarrow$ $\epsilon_{mn} = \frac{1}{2}\left(\dfrac{\partial u_m}{\partial x_n} + \dfrac{\partial u_n}{\partial x_m}\right)$ (6 Eqs)

Stress-Strain Eqs $\longrightarrow$ ϵ_{mn} = function of σ_{mn} (6 Eqs)

Shear stress $\sigma_{12} = \tau_{xy}$
Shear strain $\epsilon_{12} = \frac{1}{2}\gamma_{xy}$

Bound Conds —



$$\begin{cases} \text{on } A_1: & [\sigma_{mn}\cos(\bar{N}x_m)]\,\bar{i}_n = \bar{\sigma}_s = \sigma_{sn}\bar{i}_n \\ \text{on } A_2: & \bar{u} = u_n \bar{i}_n \end{cases}$$

prescribed stress on dA

prescribed displace

Given the Applied Loads.

Find: a) Internal Stresses

b) Deflections

c) Does it Fail? $\leftarrow$

Failure, Fracture and Fatigue

Failure of a structure means it can no longer meet its operational requirements

Modes of structural failure —

a) Material Failure (yielding, fracture)
b) Buckling
c) Loss of Stiffness (divergence, flutter, control)
d) Fatigue (repeated loads, cracks)
e) Creep (long times at high temperature)
f) Wear, Rubbing, Corrosion, Material Aging, etc

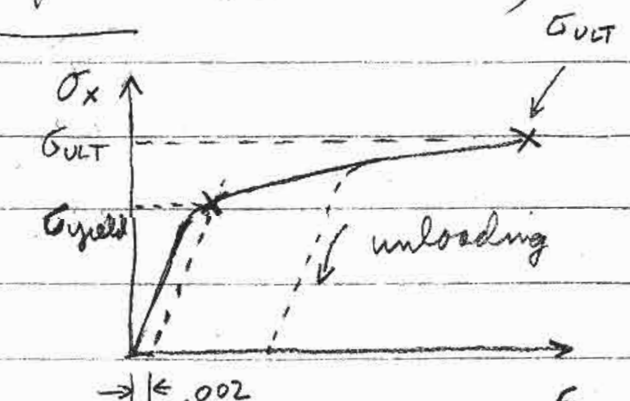
Material Failure

(Consider isotropic materials)

For bar in tension —

$\sigma_x \longleftarrow$ (————) $\longrightarrow \sigma_x$

only σ_x present
(uniaxial stress)

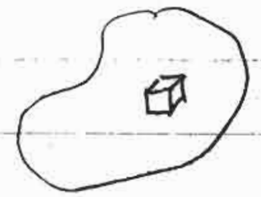


Material yields @ σ_{Yield}
" breaks @ σ_{ULT}

Permanent set for stresses above σ_Y

σ_{Yield} defined as σ for .2% permanent set
(For Metals)

For general 3-dimensional body —



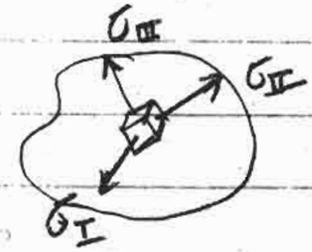
6 stresses $\sigma_x\ \sigma_y\ \sigma_z\ \tau_{xy}\ \tau_{yz}\ \tau_{zx}$
present at any point
(Combined Stress)

How adapt uniaxial test results to Combined Stress state? Several Theories proposed.

(a) <u>Maximum Normal Stress Theory</u> (Lamé)

"If max normal stress $> \sigma_{ULT}$, material breaks"

Max normal stress $\longrightarrow$ largest of the 3 principal stresses.



Principal Stresses are when:
$\begin{cases} \text{Max Normal Stresses,} \\ \text{Zero Shear Stresses} \end{cases}$

For combined stress $\sigma_x\ \sigma_y\ \sigma_z\ \tau_{xy}\ \tau_{yz}\ \tau_{zx}$, principal stresses found from condition,

$$\begin{vmatrix} (\sigma_x-\sigma) & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & (\sigma_y-\sigma) & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & (\sigma_z-\sigma) \end{vmatrix} = 0$$

Expanding gives cubic equa in σ.
The 3 roots are σ_I, σ_{II}, σ_{III} ← Princip Stresses

Material breaks if,

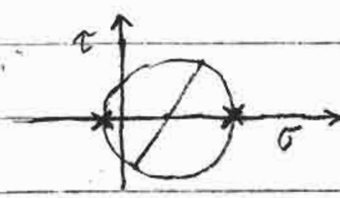
$$\boxed{\begin{array}{ll} \text{either} & \sigma_I > \sigma_{ULT} \\ \text{or} & \sigma_{II} > \sigma_{ULT} \\ \text{or} & \sigma_{III} > \sigma_{ULT} \end{array}}$$

Max Normal Stress Condition

For 2-Dimensional plane stress state —
$(\sigma_z, \tau_{xz}, \tau_{yz} = 0)$

$$\left.\begin{array}{l} \sigma_I \\ \sigma_{II} \end{array}\right\} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

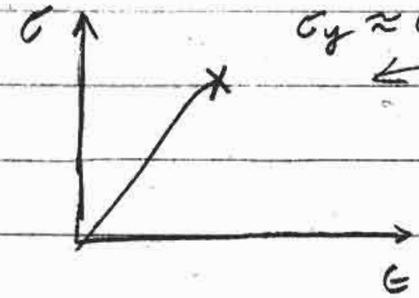
$\sigma_{III} = 0$ Mohr's Circle



So —

- Find general state of stress,
  $\sigma_x \ \sigma_y \ \sigma_z \ \tau_{xy} \ \tau_{yz} \ \tau_{zx}$
- Determine Principal Stresses σ_I, σ_{II}, σ_{III}
- Failure if σ_I, σ_{II}, $\sigma_{III} > \sigma_{ULT}$

Max Normal Stress Theory good for <u>Brittle Materials</u> only



$\sigma_y \approx \sigma_{ULT}$ ← Very little plastic range

(glass, ceramic, concrete, metals below T_g, epoxies, fibers)

<u>Note</u>: Often brittle materials have

$$(\sigma_{ULT})_{Tension} < (\sigma_{ULT})_{compress}$$

Check both tension & compression
This critical!

Max Normal Stress Theory <u>not</u> good for <u>Ductile</u> Materials

(b) Maximum Shear Stress Theory (Tresca ~1900)

"If max shear stress > σ_A, material yields"

Max shear stress → largest of the 3 quantities,

$$\frac{\sigma_I - \sigma_{II}}{2}, \quad \frac{\sigma_{II} - \sigma_{III}}{2}, \quad \frac{\sigma_{III} - \sigma_I}{2}$$

where σ_I σ_{II} σ_{III} are the principal stresses found before.

What is σ_A? For uniaxial stress (σ_x present only), yielding occurs when $\sigma_x = \sigma_{yield}$

Here $\sigma_I = \sigma_x = \sigma_{yield}$, $\sigma_{II} = \sigma_{III} = 0$

ie, Max Shear stress $= \frac{|\sigma_I - \sigma_{II}|}{2} = \sigma_A \rightarrow \boxed{\sigma_A = \frac{\sigma_{yield}}{2}}$

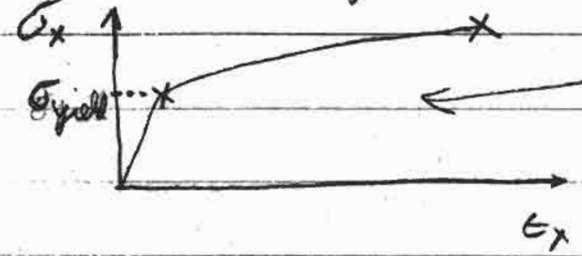
Hence, general yielding occurs if

$$\boxed{\begin{array}{ll} \text{either} & |\sigma_I - \sigma_{II}| > \sigma_{yield} \\ \text{or} & |\sigma_{II} - \sigma_{III}| > \sigma_{yield} \\ \text{or} & |\sigma_{III} - \sigma_I| > \sigma_{yield} \end{array}}$$

Max Shear Stress Condition

(Tresca Cond.)

Max Shear Stress Theory good for <u>Ductile Materials</u>
Gives onset of yield.



much plastic range (aluminum, steel, etc)

<u>Note:</u> For hydrostatic pressure,

$$\sigma_I = \sigma_{II} = \sigma_{III} = -p$$

Max Shear Cond cannot give yielding

$$|\sigma_I - \sigma_{II}| = 0 \quad \text{etc}$$

This agrees with experimental evidence.
Ductile materials can support hydrostatic pressures of 1,000,000 psi without yielding (Bridgeman experiments, Harvard, 1900's)
Yielding caused by shearing action.
(dislocations) See CDL 5.1, 5.2, 5.14

(c) Mises - Henchy Theory ($\sim$1913)

If, $\sqrt{(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2} > \sigma_B$, material yields

For uniaxial stress, yielding occurs when $\sigma_x = \sigma_I = \sigma_{yield}$. Placing into above gives,

$$\sqrt{2\,\sigma_{yield}^2} = \sigma_B \quad \text{or} \quad \sigma_B = \sqrt{2}\,\sigma_{yield}$$

Hence general yielding if,

$$\sqrt{\frac{1}{2}\left\{(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2\right\}} > \sigma_{yield}$$

or, in terms of actual stresses,

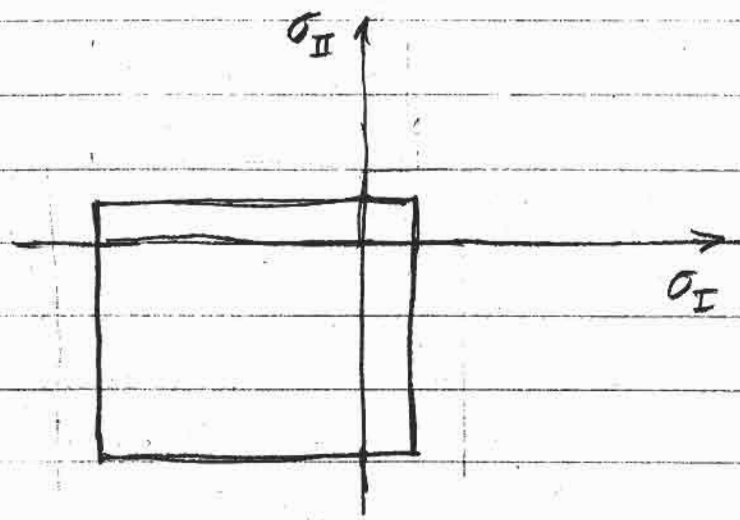
$$\boxed{\sqrt{\frac{1}{2}\left\{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2\right\} + 3\tau_{xy}^2 + 3\tau_{yz}^2 + 3\tau_{zx}^2} > \sigma_{yield}}$$

Mises - Henchy Condition

Good for Ductile Materials. Predicts onset of yield slightly better than Max Shear Stress Theory.

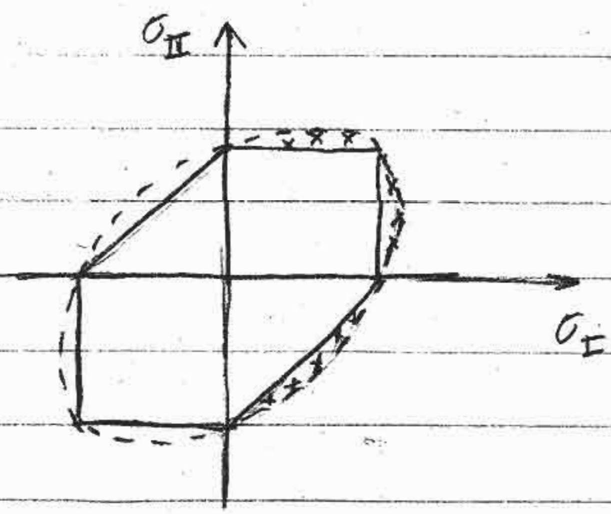
Comparison of 3 Failure Theories
for 2-Dimensional Plane Stress ($\sigma_x, \sigma_y, \tau_{xy}$ only)
Principal Stresses → $\sigma_{III} = 0$, σ_I, σ_{II} present

Brittle Materials:



Inside → O.K.
Outside → Breaks

Ductile Materials



Max Shear Stress ———
Mises Hencky - - - -

Inside → O.K.
Outside → Yield

See Crandall, Dahl, Lardner "Intro to Mech of Solids" Chap 5

Rivello Chap 3

For Orthotropic Materials, σ_{yield} not same in all directions.

wood →  ← different strengths

Can generalize Mises-Hencky to

$$\boxed{F(\sigma_y-\sigma_z)^2 + G(\sigma_z-\sigma_x)^2 + H(\sigma_x-\sigma_y)^2 + 2L\tau_{yz}^2 + 2M\tau_{zx}^2 + 2N\tau_{xy}^2 > 1}$$

"Hill Criterion"

where F, G, H, L, M, N → empirical fitted constants

However, Hill criterion has equal compressive and tensile strengths. For more generality, use

$$\boxed{C_1(\sigma_y-\sigma_z)^2 + C_2(\sigma_z-\sigma_x)^2 + C_3(\sigma_x-\sigma_y)^2 + C_4\sigma_x + C_5\sigma_y + C_6\sigma_z + C_7\tau_{yz}^2 + C_8\tau_{zx}^2 + C_9\tau_{xy}^2 > 1}$$

"Hoffman Criterion"

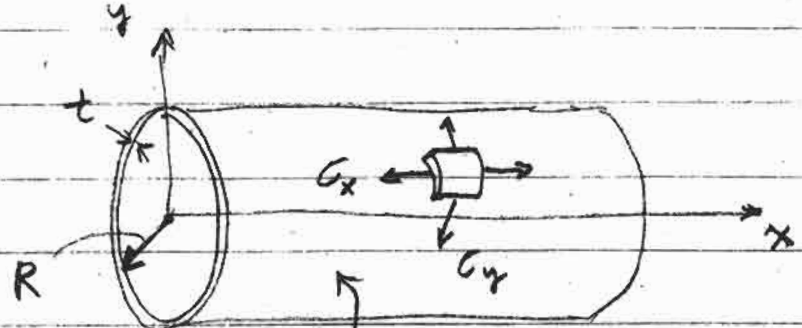
9 Terms → 3 tensile for each direction
3 compressive " " "
3 shear interaction

General Failure Analysis Procedure

1. Analyze structure for stresses σ_{ij} strains ϵ_{ij} and deflections u_i
2. Obtain yield stresses σ_{yield}, σ_{ult} from handbook (MIL-HDBK-5, etc)
3. Choose failure criterion
4. Use stresses or strains in failure criteria with σ_{yield}, σ_{ult}

Application to Pressure Tanks

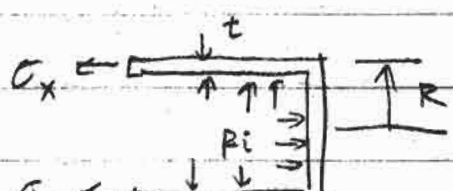
Cylindrical tank, closed @ ends, under internal pressure p_i. What p_i will cause tank to yield?



Radius R
thickness t

internal pressure p_i (lbs/in^2)

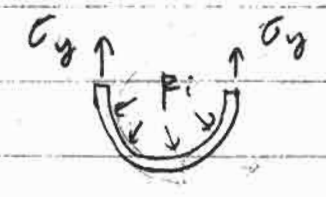
To find stresses in wall, cut $\perp$ to x axis,



$$\sum F_{Horiz} = 0 \rightarrow \sigma_x t \, 2\cancel{\pi}\cancel{R} = p_i \cancel{\pi} R^{\cancel{2}}$$

$$\sigma_x = \frac{p_i R}{2t}$$

Also cut $\parallel$ to x axis,



$$\sum F_{vert} = 0 \rightarrow \cancel{2}\sigma_y t \cancel{\ell} = p_i \cancel{2} R \cancel{\ell}$$

$$\sigma_y = p_i \frac{R}{t}$$

"Hoop stress"

No shear stress τ_{xy}

Also $\sigma_z \approx p_i$ to 0 ← negligible compared to σ_x & σ_y for $R/t > 30$

$\sigma_z \approx 0$

Plane Stress State

$$\sigma_I = \sigma_x = \frac{p_i R}{2t}$$

$$\sigma_{II} = \sigma_y = \frac{p_i R}{t}$$

$$\sigma_{III} = \sigma_z = 0$$

Apply <u>Max Shear</u> Condition —

Yield when $\left|\frac{p_i R}{2t} - \frac{p_i R}{t}\right| > \sigma_{yield}$

or $\left|\frac{p_i R}{2t}\right| > \sigma_{yield}$

or $\left|\frac{p_i R}{t}\right| > \sigma_{yield}$ ← This is critical!

Hence, for $\boxed{p_i = \sigma_{yield}\frac{t}{R}}$ ← Tank yields

Apply <u>Mises-Hencky</u> Condition

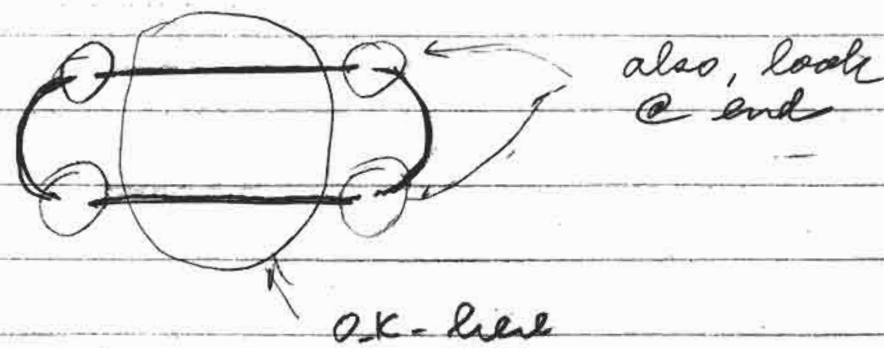
Yield when $\sqrt{\frac{1}{2}\left\{\underbrace{\left(-\frac{p_i R}{2t}\right)^2 + \left(\frac{p_i R}{t}\right)^2 + \left(-\frac{p_i R}{2t}\right)^2}_{\frac{3}{2}\left(\frac{p_i R}{t}\right)^2}\right\}} = \sigma_{yield}$

or $\frac{\sqrt{3}}{2}\frac{p_i R}{t} = \sigma_{yield}$

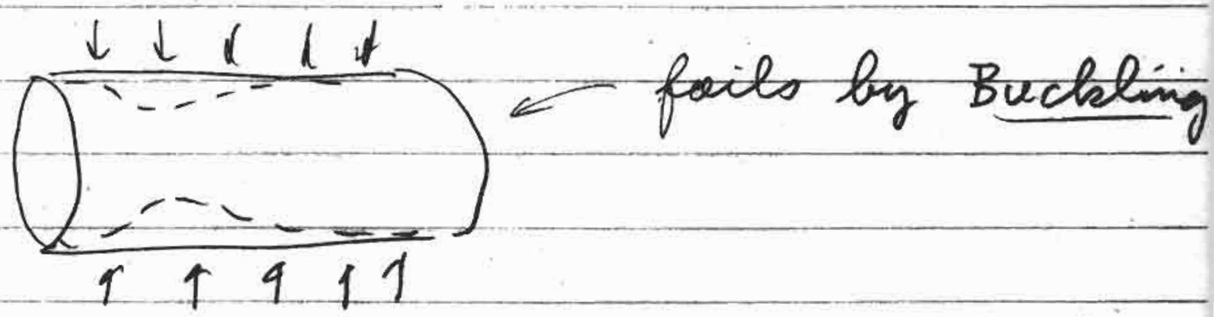
Tank yields

Hence, for $\boxed{p_i = \frac{2}{\sqrt{3}}\sigma_{yield}\frac{t}{R} = 1.15\,\sigma_{yield}\frac{t}{R}}$

Note: This is for wall only



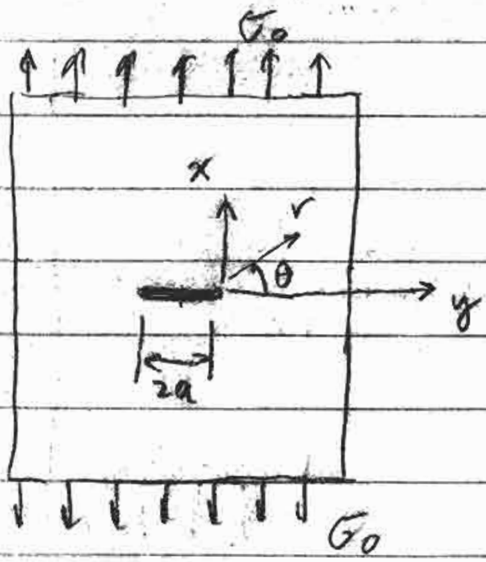
For External Pressure p_o —



Fracture Mechanics

Previous failure was for "unflawed" materials. Look now at effects of "cracks" in materials (due to manufacture, handling, fatigue, etc)

Consider plate under tension with crack



crack length $2a$
applied stress σ_o

Can show from Plane Stress Theory,

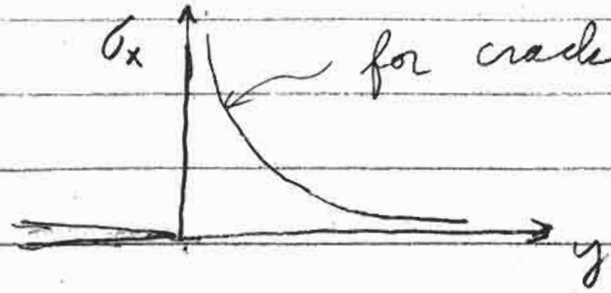
$$\sigma_x = \frac{K}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right]$$

$$\sigma_y = \frac{K}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right]$$

$$\tau_{xy} = \frac{K}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

Where $K = \sigma_o\sqrt{\pi a} \rightarrow$ "stress intensity factor"

Note Stress concentration at crack tip

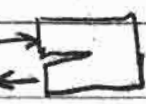


for crack, $\sigma_x \propto \frac{1}{\sqrt{r}} \to \infty$

for hole, $\sigma_x \to 3\sigma_0$

Fracture occurs when $K = K_c$ (critical stress intensity factor)
However, depends on how crack opens,

Mode I – opening  K_I

Mode II – shearing  K_{II}

Mode III – tearing  K_{III}

For this problem, have Mode I

$$K = K_I \equiv \sigma_0 \sqrt{\pi a}$$

For Fracture: $\boxed{\sigma_0 \sqrt{\pi a} = K_{Ic}}$

Griffith Equa.

$K_{Ic} \to$ "Fracture Toughness" (a material property)
$\sigma_0 \to$ Failure stress
$a \to$ crack length

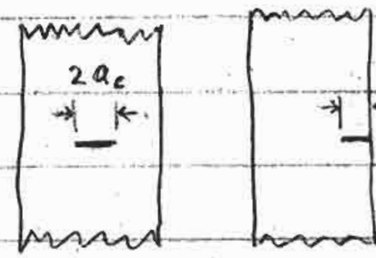
Can use to estimate failure stress σ_F given a crack size, $2a$

$$\boxed{\sigma_F = \frac{K_{IC}}{\sqrt{\frac{\pi}{2}}\sqrt{2a}}}$$

"Fast Fracture"

Also use to find critical crack size $2a_c$ for failure,

$$2a_c = \frac{2}{\pi}\left(\frac{K_{IC}}{\sigma_F}\right)^2$$



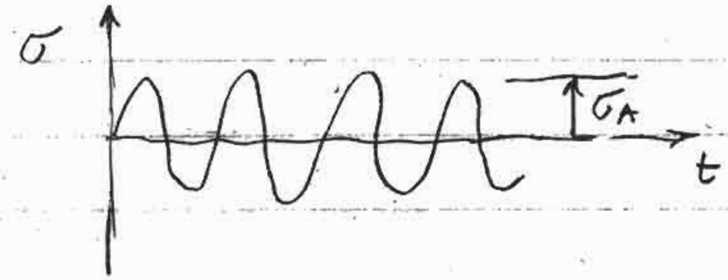
Important for Inspection technique

Typical Values —

Material	σ_{ULT} (Ksi)	σ_{TY} (Ksi)	K_{IC} (Ksi $\sqrt{in}$)	for $\sigma_F = \sigma_{ULT}$ $2a_{cr}$ (in)
Al 2024-T3	64	47	45	.31
Al 7075-T	77	69	30	.10
Ti 6A-4V	130	120	55	.11
4340 Steel	260	215	51	,02
D6ac	220	185	60	.05

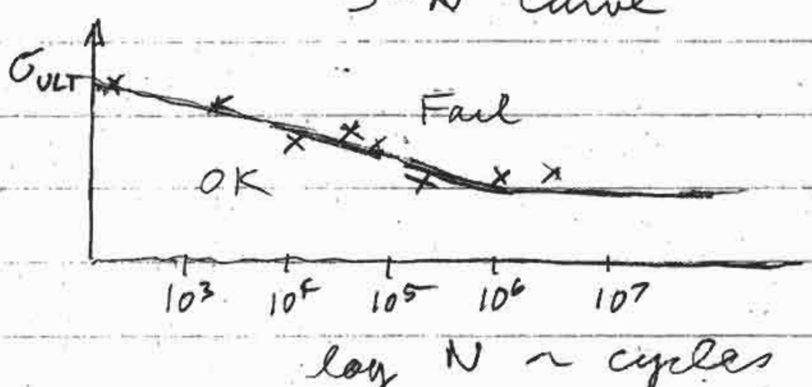
Fatigue Failure

Repeated loadings on structure



$$\sigma = \frac{P}{A}$$

"S-N Curve"



Lower σ_{max} for large N

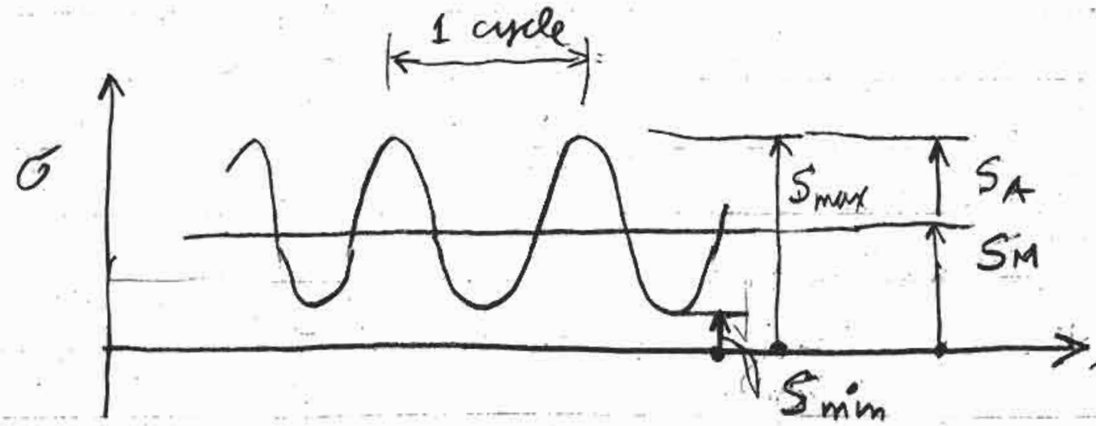
← Some materials have asymptote

σ_e → "endurance limit" or "fatigue limit"

Much scatter in data

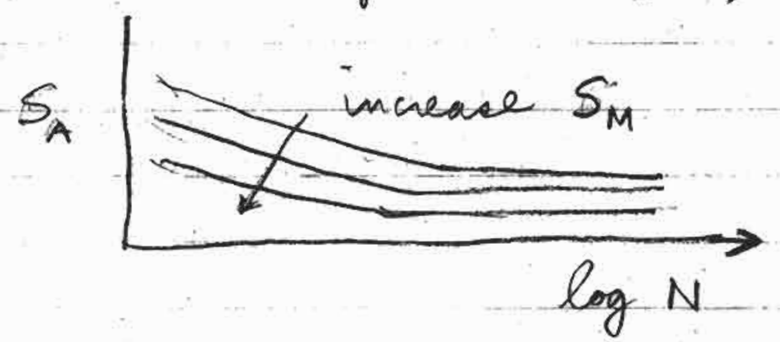
Fatigue → "Tendency of a material to break under repeated loading"

More generally, include effect of mean stress, σ_M

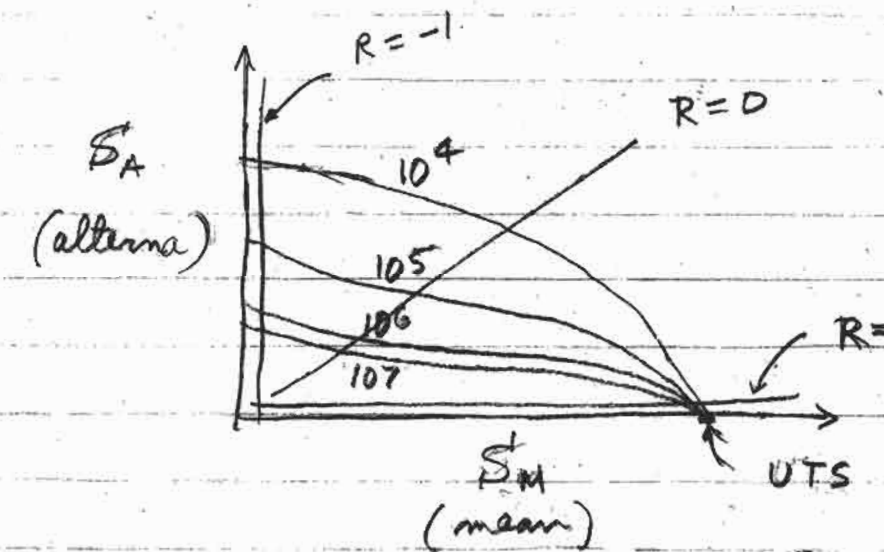


$$R = \frac{S_{min}}{S_{max}}$$

S-N Diagram becomes,



Effects better summarized in "Goodman Diagram"



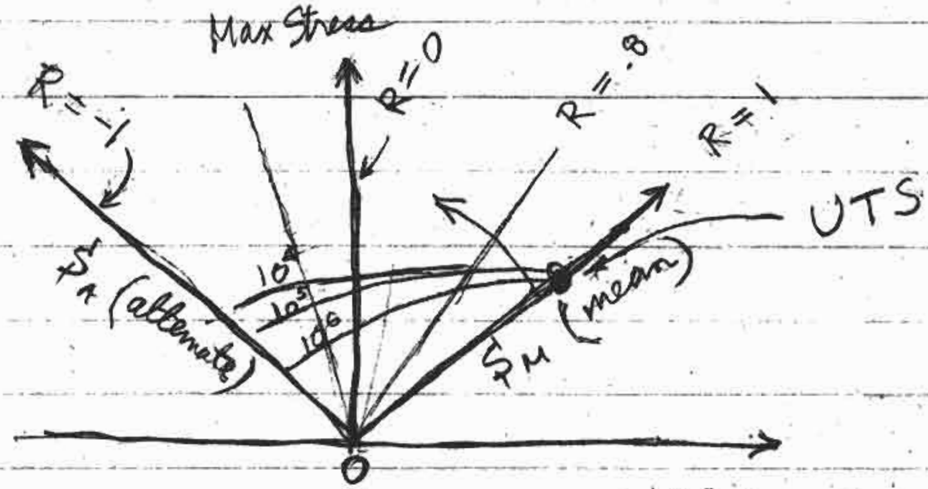
Altern, $S_A = \frac{1}{2}(S_{max} - S_{min})$

Mean, $S_M = \frac{1}{2}(S_{max} + S_{min})$

Stress Ratio, $R = \frac{S_{min}}{S_{max}}$

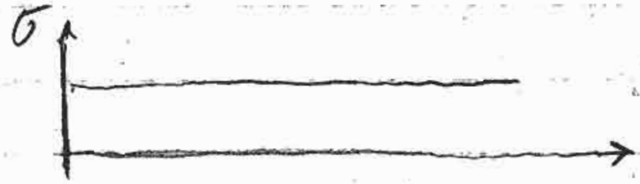
<u>See</u> MIL-HDBK-5C etc.
(TL 699)

Goodman Diagram often printed like this,

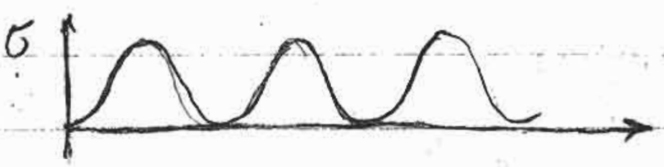


$$R = \frac{\sigma_{min}}{\sigma_{max}}$$

R = 1

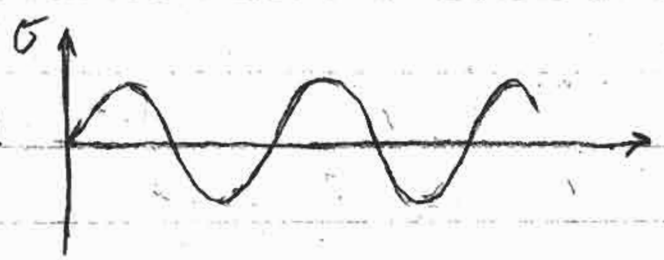


R = 0



Tension "bad"

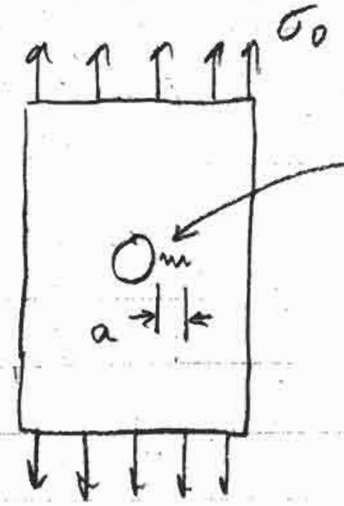
R = -1



Compress closes gaps

For Comp - Comp → little damage to material

Fatigue connected with Stress Concentrations and Crack Formation



3 × Stress here

Operating @ higher σ

$\sigma = K_T \, \sigma_0$

Stress Concentration factor (3 or 4)

Crack forms at hole.

Use Goodman diagram for $K_T = 4$ here

Different Goodman curves for different K_T's

Unnotched $K_T = 1$
Notched $K_T = 4, 5$

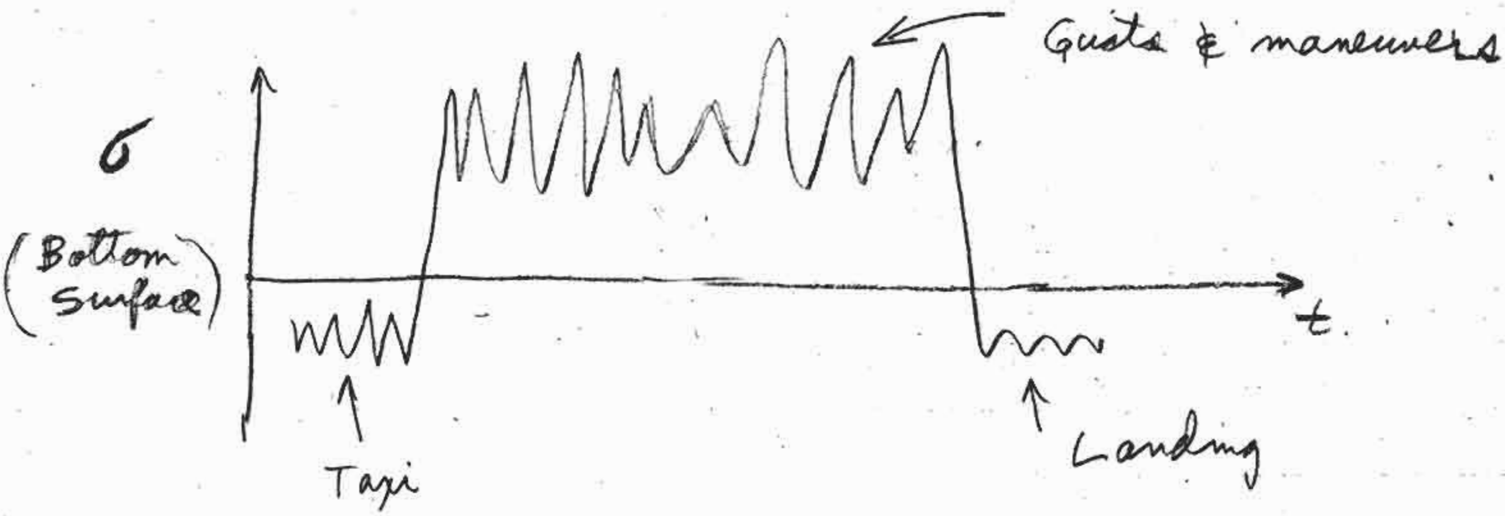
(Usually, can't multiply by K_T.)

K_I = Stress Intensity Factor $= \sigma_0 \sqrt{\pi a}$

Recall.

Cumulative Damage

For a given aircraft flight, wing stress σ may look like



Break up into M components. Two ways to estimate fatigue life,

a) Miner's Rule

b) Crack Growth Mechanics

a) Miner's Rule (~1945)

Damage, D = 0 to 1

For given cycle type #1,

$$D = \frac{n_1}{N_1}$$

n_1 = actual cycles at stress σ_1

N_1 = cycles for failure at stress σ_1

For M different cycle types,

$$D = \frac{n_1}{N_1} + \frac{n_2}{N_2} + \cdots = \sum_{i=1}^{M} \frac{n_i}{N_i}$$

Failure when,

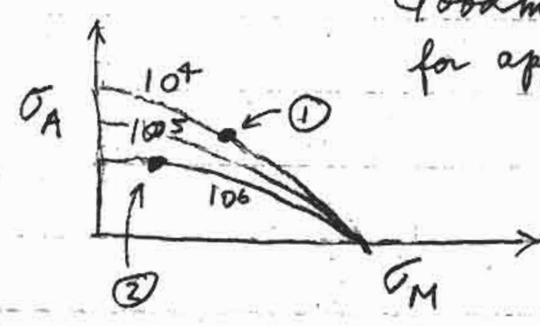
$$\boxed{\sum_{i=1}^{M} \frac{n_i}{N_i} = 1}$$ Miner's Rule

For <u>Safety</u>, compute 4 x actual life, "Scatter factor" = 4. Set <u>D = .25</u> not 1

For Example,

5,000 cycles @ $\sigma_{①}$

300,000 " @ $\sigma_{②}$



Goodman Diag for appropriate k_T

$$D = \frac{5000}{10{,}000} + \frac{300{,}000}{1{,}000{,}000} = .80 \leftarrow \underline{\text{Not good}}$$

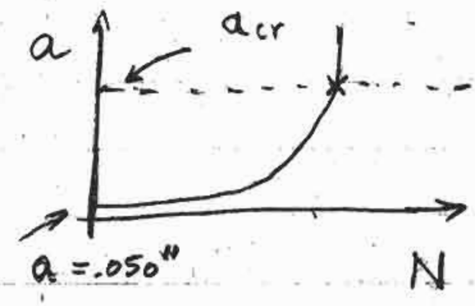
(D should be < .25)

(b) Crack Growth Fracture Mechanics

Look at crack growth,

$$\frac{da}{dN} = f(a, \sigma)$$

integrate



When crack size, $a \to a_{cr}$;
crack propagates catastrophically

Recall,

$$\boxed{K_{IC} = \alpha\, \sigma_{cr} \sqrt{\pi a_{cr}}}$$

"Fracture Toughness" — Shape factor ~ 1 — Critical Stress — critical crack size

(Originally, High strain a measure of toughness)

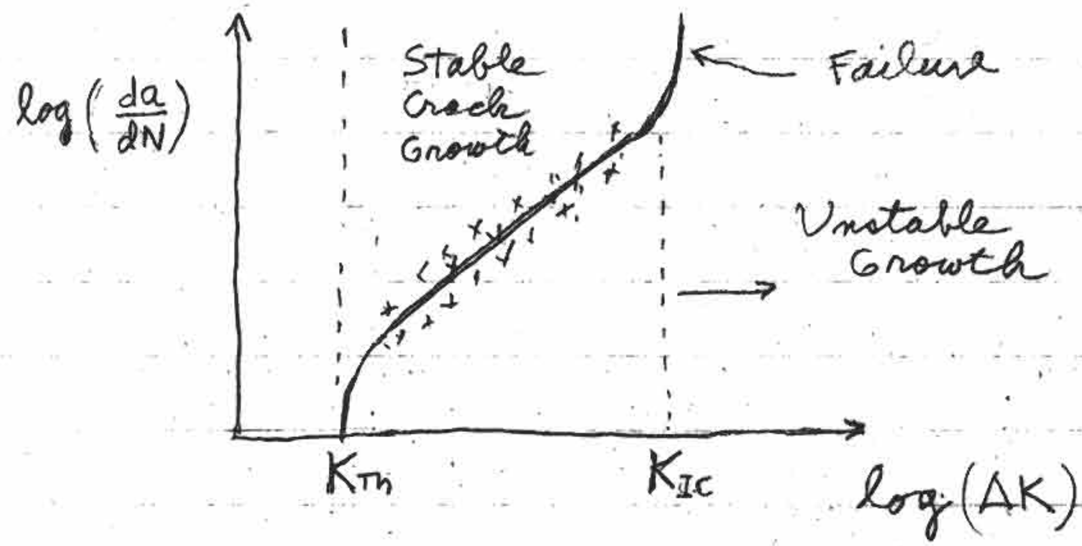
High $K_{IC} \rightarrow$ large a_{cr} for given σ_{cr}

Introduce, Stress Intensity Factor Range ΔK,

$$\Delta K = K_{max} - K_{min} = \alpha(\sigma_{max} - \sigma_{min})\sqrt{\pi a}$$

Note: if $\sigma_{min} < 0$, set $K_{min} = 0$ ← ignore compress

Can plot crack growth rate da/dN versus stress intensity factor range ΔK



Middle region is straight line on log-log plot

Paris Law for Stable Crack Growth,

$$\boxed{\frac{da}{dN} = C\left(\Delta K\right)^m}$$

$$\Delta K = \alpha \, \Delta\sigma \sqrt{\pi a}$$

C, m → constants

$m \approx 3$ for steel

$\approx 3 - 4$ for alum

N = Cycles

Obtain cycles to failure N_F by integrating Paris Law

$$\frac{da}{dN} = C\left(\alpha\,\Delta\sigma\sqrt{\pi a}\right)^m$$

$$N_F = \int_0^{N_F} dN = \frac{1}{C\pi^{m/2}\Delta\sigma^m}\int_{a_0}^{a_{cr}} \frac{da}{\alpha^m\, a^{m/2}}$$

If assume α independent of a, then

$$N_F = \frac{a_{cr}^{\left(-\frac{m}{2}+1\right)} - a_0^{\left(-\frac{m}{2}+1\right)}}{C\pi^{m/2}\alpha^m\,\Delta\sigma^m\left(-\frac{m}{2}+1\right)}$$

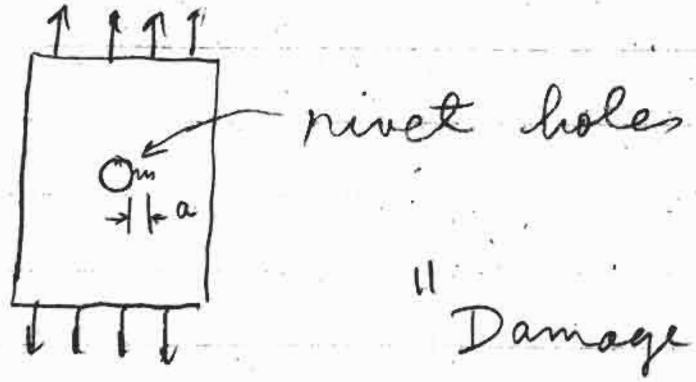
where, $a_{cr} = \frac{1}{\pi}\left(\frac{K_{Ic}}{\alpha\,\sigma_{max}}\right)^2$

Here, N_F depends on assumed a_0 (choose biggest estimate)

Can also modify Paris Law for mean stress effects —

$$\frac{da}{dN} = \frac{C(\Delta K)^m}{(1-R)K_c - \Delta K}$$ "Forman's Law"

<u>Inspect</u> periodically for crack sizes in critical locations -



"Damage Tolerance"

Fracture Mechanics vs. S-N curves

See Ashby & Jones, "Eng'g Materials 1" Chaps 13 - 16

For Overall Structure, recall also other modes of Failure —

Buckling, Loss of Stiffness, Creep, Wear, Corrosion, Aging, etc.

Design Approaches to Longevity

1. Infinite Life Design
   (Use $\sigma < \sigma_{endurance\ limit}$)
   (buildings, valves)

2. Safe-Life Design
   (Estimate life, and throw away after)
   (helicopter blades, turbine rotors ?)

3. Fail-Safe Design
   (Use redundant paths, so failure is not catastrophic)
   (alternate controls)

4. Damage Tolerant Design
   (Check that cracks don't grow to a critical size. Inspection!)

5. Empirical Test
   (Make part and test several cyclicly to failure). (testing land gear)

For Aircraft Structural Design,

Analysis + Test together.
Later, continue inspection
for cracks in critical locations
(rivet holes, corners, ...) and for
corrosion, wear, etc

Ease of inspection & replacement.

(Automobile examples)

Aircraft → Comet, DC-10 Pylon,