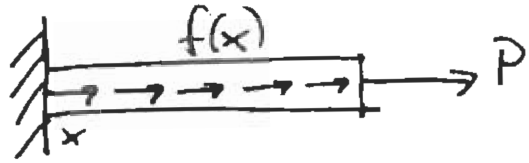
The Ritz Method (cont'd)

$$\frac{d}{dx}\left(EA\,u'\right) = f(x) \qquad 0 < x < L \qquad \text{(Model problem)}$$

Potential: $$\Pi = \frac{1}{2}\int_0^L AE\,u'^2\,dx - \int_0^L f\,u\,dx$$

Boundary conditions:



$u(0) = 0$ (essential) $\qquad EA\,u'(L) = P$ (natural

Ritz Approximation: $\quad u \sim c_i\,\phi_i(x) \qquad i = 1, N$

$$\Pi(u) \sim \Pi(c_i\,\phi_i) = \Pi(c_i)$$

Equilibrium: $$\delta\Pi = \frac{\partial \Pi}{\partial c_i}\,\delta c_i = 0$$

$\Longrightarrow$ $N \times N$ system of equations to determine c_i

$$\Pi = \frac{1}{2}\int_0^L EA\,(c_i\,\phi_i')^2\,dx - \int_0^L f\,c_i\,\phi_i\,dx$$

$$\frac{\partial\Pi}{\partial c_i} = \underbrace{\int_0^L EA\,\phi_i'\,\phi_j'\,dx}_{K_{ij}}\;c_i - \underbrace{\int_0^L f\,\phi_i\,dx}_{R_i}$$

Solve for c_i : $[K]\{c\} = \{R\}$

$$\{c\} = [K]^{-1}\{R\}$$

Reconstruct approximate solution from obtained coefficients.

<u>Alternative formulation using PVD:</u>

PVD: $\int_0^L \sigma\,\delta\varepsilon\,A\,dx = \int_0^L f\,\delta u\,dx \quad \forall$ admissible δu

$\sigma = E u'$, approximate $u \sim c_i\,\phi_i$, as usual

Approximate the virtual displacements as:

$\delta u = \delta c_i\,\phi_i$, then

$\delta\varepsilon = \delta c_i\,\phi_i'$

$$\int_0^L EA\, c_i \phi_i' \phi_j' \, dx \, \delta c_j = \int_0^L f \phi_k \, dx \, \delta c_k$$

$$\text{or} \quad \Big[\underbrace{\int_0^L EA\, \phi_j' \phi_i' \, dx}_{K_{ij}} \, c_j - \underbrace{\int_0^L f \phi_i \, dx}_{R_i} \Big] \delta c_i = 0 \qquad \forall \delta c_i$$

or $[K]\{C\} = \{R\}$ as before.

Conditions on ϕ_i's

1. • Want approximate solutions that become closer to exact solution "u" as "N" is increased

2. • Want ϕ_i such that conditions of PVD are satisfied.

3. • Want system $[K]\{C\} = \{R\}$ to have unique solution (linearly independent equations).

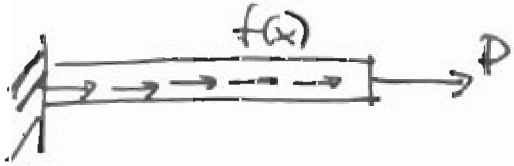
- ϕ_i some continuity such that integrals in K_{ij} can be evaluated (exist, $< \infty$)
- satisfy the essential boundary conditions

if $u = \bar{u} \neq 0$ on some part of S_u
we satisfy this by requiring:

$\phi_1(x) = \bar{u}$ and all others $= 0$ on this part of the boundary.

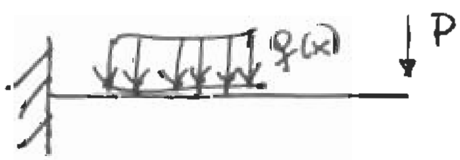
- the set of functions $\{\phi_i\}$ must be "complete"

These conditions do not provide guidelines for generating the functions.
Usually one adopts a family of simple functions (polynomials, trigonometric functions) satisfying the requirements above.

- 

- $\phi_i = x^i$
- $\phi_i = \sin\left[\frac{(2i+1)\pi x}{2L}\right]$

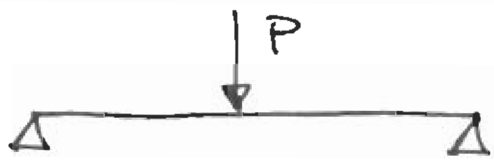
note that $\phi_i = \sin \frac{\pi i x}{L}$ would give $U(L) = 0$

- 

$w_0 \sim C_i \phi_i$, $\phi_i = \sin\left[(2i+1)\frac{\pi}{2}\frac{x}{L}\right]$

$\phi_i = x^i$

Convergence could be very slow for a poor choice of basis functions ϕ_i

- 

solution is two piecewise cubic polynomials.

$\phi_i = \sin \frac{\pi i x}{L}$ gives very slow convergence.

Remarks:

- for well-chosen ϕ_i's the process converges (proof omitted)
- for increasing "N" the previously computed "c_i's" don't change.
- K_{ij} is symmetric for linear elasticity
- strains and stresses are generally
- governing equation and natural boundary condition satisfied in the variational (integral) sense. Therefore the equation of equilibrium is not satisfied pointwise.
- ~~since a continuous system (∞ degrees of freedom) is approximated with a finite number~~
- from PMPE, approximate solution minimizes energy within subspace of functions $\Rightarrow$ not the real minimum $\Rightarrow$ energy is higher $\Rightarrow$ system is stiffer.