Perturbation Methods / Overview

1 OVERVIEW

1. Regular/straight perturbation method
   a) Systematically generates approximations order by order
   b) Proceeds to higher degree of approximations by generating corrections of higher-order in the expansion parameter, ϵ
   c) Breaks down when higher-order perturbation solutions fail (less accuracy) in a domain of interest: nonuniformity of the perturbation expansion
2. Types of nonuniformities
   a) Secular - nonuniformity occurs as $t \to \infty$
   b) Singular - nonuniformity occurs as $t \to 0$
   c) Boundary layer - systems, resulting in a loss of boundary conditions

2 SECULAR NONUNIFORMITY

Approximations break down (inaccurate) for large values of t.

Example:

$$\frac{dx}{dt} + \epsilon x = 0, x(0) = 1, 0 < \epsilon << 1$$

Apply regular perturbation theory and obtain:

$$x(t;\epsilon) = x_0(t) + \epsilon x_1(t) + ...$$

$$\frac{dx_0}{dt} = 0$$

$$\frac{dx_1}{dt} = -x_0$$

$$\frac{dx_n}{dt} = -x_{n-1}$$

Solve recursively

$$x(t;\epsilon) = 1 - \epsilon t + \frac{\epsilon^2 t^2}{2!} - ... + (-1)^n \frac{\epsilon^n t^n}{n!} + ...$$

For a complete, convergent, summable infinite series we may write:

$$x(t;\epsilon) = e^{\epsilon t}$$

Which is the exact solution to the above problem. If summation is not possible (general term cannot be determined), we truncate series and write:

$$x(t;\epsilon) = \sum_{0}^{n} \epsilon^i x_i(t) = \sum_{i=0}^{n} (-1)^i \frac{\epsilon^i t^i}{i!}$$

Problems!

So long as $t = O(1)$ or less

$$x(t;\epsilon) = 1 - \epsilon t + \frac{\epsilon^2 t^2}{2!} - ...$$

Is a good approximation. But for $t \geq O(\frac{1}{\epsilon})$, accuracy begrades as n increases. Hence $x(t;\epsilon) = x_0(t) + \epsilon x(t) + ...$ is secularly nonuniform [it breaks down for large values of t]. Use multiple time scales method to solve this problem.

3 Singular perturbation

A. Lighthill type

- Higher order perturbation terms are more singular than lower order perturbation terms.

Consider:

$$(t+\epsilon)\frac{dx}{dt}+x=0, x(1)=1$$

Substitute:

$$x(t;\epsilon)=x_0(t)+\epsilon x_1(t)+...$$

Obtain:

$$t\frac{dx_0}{dt}+x_0=0$$

$$t\frac{dx_1}{dt}+x_1=-\frac{dx_0}{dt}$$

$$t\frac{dx_n}{dt}+x_n=-\frac{dx_{n-1}}{dt}$$

Zeroth order solution:

$$x_0(t)=\frac{1}{t}$$

First order solution:

$$x_1(t)=\frac{1}{t}-\frac{1}{t^2}$$

Hence to $O(\epsilon)$

$$x(t;\epsilon)\cong\frac{1}{t}+\epsilon(\frac{1}{t}-\frac{1}{t^2})$$

Problems!

$x_0(t)$ is singular at $t=0$
$x_1(t)$ is more singular at $t=0$
$x_1(t)$ is better than $x_0(t)$ for large values of t

Exact solution:

$$x(t;\epsilon)=\frac{1+\epsilon}{t+\epsilon}$$

Expanding above for large values of t:

$$x(t;\epsilon)=\frac{1+\epsilon}{t}(1-\frac{\epsilon}{t}+...)$$

B. Boundary-layer type

- Inability to satisfy initial or boundary conditions in the limit

Consider:

$$\epsilon \frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0, x(0) = 0, x(1) = 1$$

Substitute:

$$x(t;\epsilon) = x_0(t) + \epsilon x_1(t) + \dots$$

Obtain:

$$\frac{dx_0}{dt} + x_0 = 0$$

$$\frac{dx_1}{dt} + x_1 = -\frac{d^2x_0}{dt^2}$$

Obtain (solving order by order):

$$x_0(t) = c_1 e^{-t}$$

$$x_1(t) = -c_1 te^{-t} + c_2 te^{-t}$$

$$x(t;\epsilon) \cong c_1 e^{-t} + \epsilon(-c_1 te^{-t} + c_2 e^{-t}) + \dots$$

c_1 and c_2 are arbitrary constants

Problems!

$x(0) = 0$ implies $c_1 = 0$ from zeroth-order solution.

Since only one arbitrary constant is available to choose at zeroth order, it is not possible to satisfy both boundary conditions.

The characteristic equation of:

$$\epsilon \frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0$$

is

$$\epsilon s^2 + s + 1 = 0$$

Hence

$$S_{1,2} = \frac{-1 \pm \sqrt{1-4\epsilon}}{2\epsilon}$$

Therefore, the exact solution is

$$x(t;\epsilon) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

Expanding square root to obtain:

$$s_1 \sim -\frac{1}{\epsilon} + \ldots$$

$$s_2 \sim 1 + \ldots$$

And asymptotically

$$x(t;\epsilon) \cong c_1 e^{-\frac{t}{\epsilon}+\cdots} + c_2 e^{-t+\cdots}$$

$c_1 e^{-\frac{t}{\epsilon}+\cdots}$ is the fast, inner solution; boundary solution
$c_2 e^{-t+\cdots}$ is the slow, outer solution

4 Poincare-Lighthill Method: Method of strained coordinates

- Represent the independent variable as a nonuniform series in powers of the small parameter, ϵ

Consider

$$(t+\epsilon)\frac{dx}{dt} + x = 0, x(1) = 1$$

Recall problems generated by applying regular perturbation methods.

Lighthill suggested:

$$t = s + \epsilon t_1(s) + \epsilon^2 t_2(s) + \ldots$$

The "stretching" functions $t_1(s), t_2(s), \ldots, t_n(s)$ are to be determined. Now determine $\frac{d}{dt}$:

$$\frac{d}{dt} \to \frac{d}{ds}\frac{ds}{dt} = \frac{1}{dt/ds}\frac{d}{ds} = \frac{1}{(1+\epsilon\frac{dt_1}{ds}+\ldots)}\frac{d}{ds}$$

Can write

$$x(s:\epsilon) = x_0(s) + \epsilon x_1(s) + ...$$

Substitute and re-write transformed governing equation:

$$[s + \epsilon t_1(s) + ... + \epsilon]\frac{d}{ds}[x_0(s) + \epsilon x_1(s) + ...] + (1 + \epsilon\frac{dt_1}{ds} + ...)[x_0(s) + \epsilon x_1(s) + ...]$$

Equating like powers of ϵ_1 to first order:

$$sx_0' + x_0 = 0$$

$$sx_1' + x_1 = -[(t_1 + 1)x_0' + t_1' x_0]$$

$$x_n' \to \frac{dx_n}{ds}$$

Zeroth order solution

$$x_0(s) = \frac{c_1}{s}$$

$$c_1 = \text{arbitrary constant}$$

Note $x_0(s)$ is singular at $s = 0$

$x(s:\epsilon)$ is uniform as $\to 0$, provided $x_1(s)$ is no more singular than $x_0(s)$. That is, as $s \to 0$, $x_1(s)$ cannot be singular like $\frac{1}{s^\alpha}, \alpha > 1$. This means that higher-order perturbation solutions are no more singular than lower-order ones. Hence,

$$x_0 t_1' + (t_1 + 1)x_0' = 0$$

Solving for $t_1(s)$ we have

$$t_1 = -1$$

and

$$sx_1' + x_1 = -[(t_1 + 1)x_0' + t_1' x_0]$$

Takes the form

$$\frac{d}{ds}(sx_1) = -\frac{d}{ds}[x_0(t_1 + 1)]$$

Integrating

$$sx_1 = -x_0(t_1 + 1) + c_2$$

or

$$x_1(s) = -\frac{x_0}{s}(t_1+1) + \frac{c_2}{s}$$

And $x_0 = 1/s$ to satisfy the boundary conditions. Thus, unless $t_1 = -1$, $x_1(s)$ will be more singular than $x_0(s)$. With $t_1 = -1$, we have

$$t = s + \epsilon t_1(s) + \ldots = s - \epsilon$$

so

$$t = s - \epsilon$$

or

$$s = t + \epsilon$$

Hence, substituting

$$x(t;\epsilon) \sim \frac{c_1}{s}|_{s(t)} = \frac{c_1}{t+\epsilon}$$

c_1 = arbitrary constant

Note the uniformity condition at the first order led to the correct change of variables and the exact solution. This procedure can be applied at each order as required.