Lecture L9 - Linear Impulse and Momentum. Collisions

In this lecture, we will consider the equations that result from integrating Newton's second law, $\boldsymbol{F} = m\boldsymbol{a}$, in time. This will lead to the principle of linear impulse and momentum. This principle is very useful when solving problems in which we are interested in determining the global effect of a force acting on a particle over a time interval.

Linear Momentum

We consider the curvilinear motion of a particle of mass, m, under the influence of a force $\boldsymbol{F}$. Assuming that the mass does not change, we have from Newton's second law,

$$\boldsymbol{F} = m\boldsymbol{a} = m\frac{d\boldsymbol{v}}{dt} = \frac{d}{dt}(m\boldsymbol{v}) \ .$$

The case where the mass of the particle changes with time (e.g. a rocket) will be considered later on in this course. The *linear momentum* vector, $\boldsymbol{L}$, is defined as

$$\boldsymbol{L} = m\boldsymbol{v} \ .$$

Thus, an alternative form of Newton's second law is

$$\boldsymbol{F} = \dot{\boldsymbol{L}} \ , \tag{1}$$

which states that the total force acting on a particle is equal to the time rate of change of its linear momentum.

Principle of Linear Impulse and Momentum

Imagine now that the force considered acts on the particle between time t_1 and time t_2. Equation (1) can then be integrated in time to obtain

$$\int_{t_1}^{t_2} \boldsymbol{F} \, dt = \int_{t_1}^{t_2} \dot{\boldsymbol{L}} \, dt = \boldsymbol{L}_2 - \boldsymbol{L}_1 = \Delta\boldsymbol{L} \ . \tag{2}$$

Here, $\boldsymbol{L}_1 = \boldsymbol{L}(t_1)$ and $\boldsymbol{L}_2 = \boldsymbol{L}(t_2)$. The term

$$\boldsymbol{I} = \int_{t_1}^{t_2} \boldsymbol{F} \, dt \ = \Delta\boldsymbol{L} = (m\boldsymbol{v})_2 - (m\boldsymbol{v})_1,$$

is called the *linear impulse*. Thus, the linear impulse on a particle is equal to the linear momentum change $\Delta\boldsymbol{L}$. In many applications, the focus is on an impulse modeled as a large force acting over a small time. But

in fact, this restriction is unnecessary. All that is required is to be able to perform the integral $\int_{t_1}^{t_2} \boldsymbol{F}\ dt$. If the force is a constant F, then $\Delta \boldsymbol{L} = \int_{t_1}^{t_2} \boldsymbol{F}\ dt = F(t_2 - t_1)$. If the force is given as a function of time, then $\Delta \boldsymbol{L} = \int_{t_1}^{t_2} \boldsymbol{F}(t)\ dt$


Note   **Units of Impulse and Momentum**

It is obvious that linear impulse and momentum have the same units. In the SI system they are N · s or kg · m/s, whereas in the English system they are lb · s, or slug · ft/s.

Example (MK)   **Average Drag Force**

The pilot of a 90,000-lb airplane which is originally flying horizontally at a speed of 400 mph cuts off all engine power and enters a glide path as shown where $\beta = 5^o$. After 120 s, the airspeed of the plane is 360 mph. We want to calculate the magnitude of the time-averaged drag force.
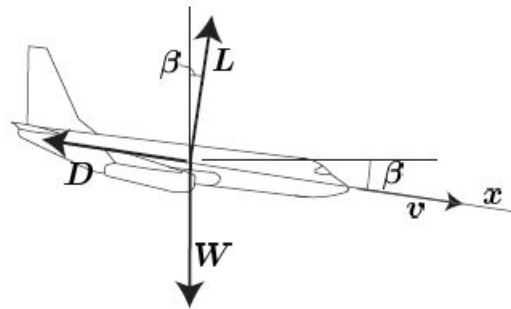
Aligning the x-axis with the flight path, we can write the x component of equation (2) as follows

$$\int_0^{120} (W \sin \beta - D)\ dt = L_x(120) - L_x(0)\ .$$

The time-averaged value of the drag force, $\bar{D}$, is

$$\bar{D} = \frac{1}{120} \int_0^{120} D\ dt\ .$$

Therefore,

$$(W \sin \beta - \bar{D})120 = m(v_x(120) - v_x(0))\ .$$

Substituting and applying the appropriate unit conversion factors we obtain,

$$(90,000 \sin 5^o - \bar{D})120 = \frac{90,000}{32.2}(360 - 400)\frac{5280}{3600} \quad \rightarrow \quad \bar{D} = 9,210\ \text{lb}\ .$$

Impulsive Forces

We typically think of impulsive forces as being forces of very large magnitude that act over a very small interval of time, but cause a significant change in the momentum. Examples of impulsive forces are those generated when a ball is hit by a tennis racquet or a baseball bat, or when a steel ball bounces on a steel plate. The table below shows typical time intervals over which some of these impulses occur.
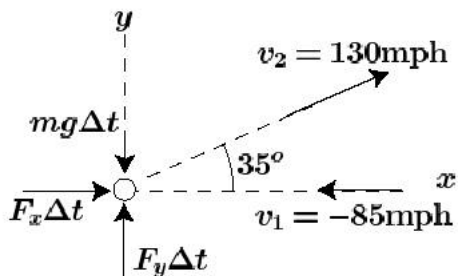
	Time interval Δt [s]
Racquet hitting a tennis ball	$0.005 - 0.05$
Bat hitting a baseball	0.01–0.02
Golf club hitting a golf ball	0.001
Pile Driver	$0.01 - 0.02$
Shotgun	0.001
Steel ball bouncing on steel plate	0.0002

Example (MK)   *Baseball Bat*

A baseball is traveling with a horizontal velocity of 85 mph just before impact with the bat. Just after the impact, the velocity of the $5\frac{1}{8}$ oz. ball is 130 mph, at an angle of 35° above the horizontal. We want to determine the horizontal and vertical components of the average force exerted by the bat on the baseball during the 0.02 s impact.

If we consider the time interval between the instant the baseball hits the bat, t_1, and the instant after it leaves the bat, t_2, the forces responsible for changing the baseball's momentum are gravity and the contact forces exerted by the bat. We can use the x and y components of equation (2) to determine the average force of contact. First, consider the x component,

$$mv_{x1} + \int_{t_1}^{t_2} F_x \, dt = mv_{x2} \ .$$

Inserting numbers and using appropriate conversion factors, we obtain

$$-\frac{5.125/16}{32.2}(85\frac{5280}{3600}) + \bar{F}_x(0.02) = \frac{5.125/16}{32.2}(130\frac{5280}{3600}\cos 35^o) \quad \rightarrow \quad \bar{F}_x = 136.7 \, \text{lb} \ .$$

For the y direction, we have

$$mv_{y1} + \int_{t_1}^{t_2} F_y \, dt = mv_{y2} \ ,$$

which gives,

$$0 + \bar{F}_y(0.02) - \frac{5.125}{16}(0.02) = \frac{5.125/16}{32.2}(130\frac{5280}{3600}\sin 35^o) \ \rightarrow \ \bar{F}_y = 54.7 \text{ lb} \ .$$

We note that $mg\Delta t = 0.0064$ lb-s, whereas $\bar{F}_y(\Delta t) = 1.094$ lb - s. Thus, $mg\Delta t$ is 0.59% of the total impulse. We could have safely neglected it. In this case we would have obtained $\bar{F}_y = 54.4$ lb.

Conservation of Linear Momentum

We see from equation (1) that if the resultant force on a particle is zero during an interval of time, then its linear momentum $\boldsymbol{L}$ must remain constant. Since equation (1) is a vector quantity, we can have situations in which only some components of the resultant force are zero. For instance, in Cartesian coordinates, if the resultant force has a non-zero component in the y direction only, then the x and z components of the linear momentum will be conserved since the force components in x and z are zero.

Consider now two particles, m_1 and m_2, which interact during an interval of time. Assume that interaction forces between them are the only unbalanced forces on the particles. Let $\boldsymbol{F}$ be the interaction force that particle m_2 exerts on particle m_1. Then, according to Newton's third law, the interaction force that particle m_1 exerts on particle m_2 will be $-\boldsymbol{F}$. Using expression (2), we will have that $\Delta \boldsymbol{L}_1 = -\Delta \boldsymbol{L}_2$, or $\Delta \boldsymbol{L} = \Delta \boldsymbol{L}_1 + \Delta \boldsymbol{L}_2 = \boldsymbol{0}$. That is, the changes of momentum of particles m_1 and m_2 are equal in magnitude and opposite in sign, and the total momentum change equals zero. Recall that this is true if the only unbalanced forces on the particles are the interaction forces. The more general situation in which external forces can be present will be considered in future lectures.

We note that the above argument is also valid in a componentwise sense. That is, when two particles interact and there are no external unbalanced forces along a given direction, then the total momentum change along that direction must be zero.


Example   *Ballistic Pendulum*

The ballistic pendulum is used to measure the velocity of a projectile by observing the maximum angle $\theta_{\max}$ to which the box of sand with the embedded projectile swings. Find an expression that relates the initial velocity of a projectile v_0 of mass m to the maximum angle $\theta_{\max}$ reached by the pendulum. The mass of the sand box is M and the length of the pendulum is L.
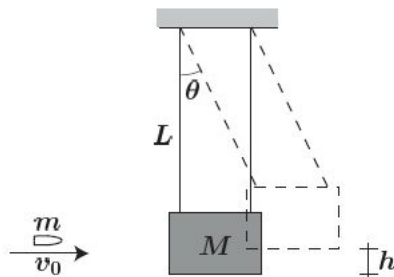
We consider the equation of conservation of linear momentum along the horizontal direction. The initial momentum of the projectile is mv_0. Since the sand box is initially at rest its momentum is zero. Just after the projectile penetrates into the box, the velocity of the sand box and the projectile are the same. Therefore, if v_1 is the velocity of the sand box (with the embedded projectile) just after impact, we have from conservation of momentum,

$$mv_0 = (M + m)v_1 .$$

After impact, the problem reduces to that of a simple pendulum. The only force doing any work is gravity and therefore we can apply the principle of conservation of work and energy. At the point when θ is maximum, the velocity will be zero. From energy conservation we finally obtain,

$$\frac{1}{2}(M + m)v_1^2 = (M + m)gh_{\max} ,$$

and since $h = L(1 - \cos\theta)$, we have

$$\theta_{\max} = \cos^{-1}\left(1 - \left(\frac{m}{M+m}\right)^2 \frac{v_0^2}{2Lg}\right) .$$

Note that energy is not conserved in the collision. The initial kinetic energy of the system is the kinetic energy of the projectile $T_0 = \frac{1}{2}mv_0^2$ (taking the reference height as zero). After the collision the kinetic energy of the system is

$$T_1 = \frac{1}{2}(m + M)v_1^2. \tag{3}$$

Since $v_1 = v_0 \frac{m}{(m+M)}$ we have

$$T_1 = \frac{1}{2}\frac{m^2}{(M + m)}v_0^2 \tag{4}$$

which is less than T_0 for $M > 0$.

Collisions

We consider now, the situation of two isolated particles colliding. The only forces on the particles are due to their mutual interaction. When the velocity vector of the two particles is parallel to the line joining the two particles, we say that we have one-dimensional collisions, otherwise we say that the collision is oblique.

1D Collisions

Here, we are dealing with rectilinear motion and therefore the velocity vector becomes a scalar quantity. In order to understand the collision process we consider five different stages.

I) **Before Impact**

Let particle 1, of mass m_1, occupy position x_1 and travel with velocity v_1 along the direction parallel to the line joining the two particles. And let particle 2, of mass m_2, occupy position x_2 and travel with speed v_2 also in the same direction. We assume that $v_1 > v_2$ so that collision will occur.
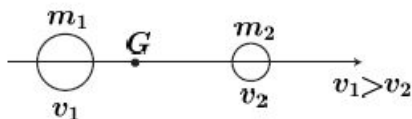

We can introduce the position of the center of mass $x_G = (m_1 x_1 + m_2 x_2)/m$, where $m = m_1 + m_2$. The velocity of the center of mass is then given by $v_G = (m_1 v_1 + m_2 v_2)/m$. Note that since conservation of momentum requires $(m_1 v_1 + m_2 v_2) = (m_1 v_1' + m_2 v_2')$, the velocity of the center of mass remains unchanged by the collision process between the particles. If we define the relative velocity $g = v_1 - v_2$ we can express v_1 and v_2 as a function of v_G and g as,

$$
\begin{aligned}
v_1 &= v_G + \frac{m_2}{m} g \\
v_2 &= v_G - \frac{m_1}{m} g \ .
\end{aligned}
$$

II) **Deformation**

The particles establish contact and the force between them F_d increases until the instant of maximum deformation.
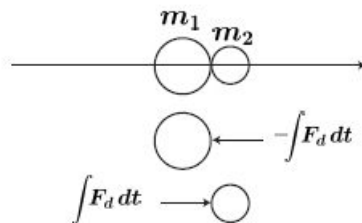
III) **Maximum Deformation**

The contact force is at its maximum and the two particles travel at the same velocity v_G. Thus the deformation force has slowed m_1 down to a velocity of v_G and sped up m_2 to a velocity of v_G.

The impulse applied to particle 1 and 2 from the deformation force equals the change in momentum in the deformation process.

For particle 1

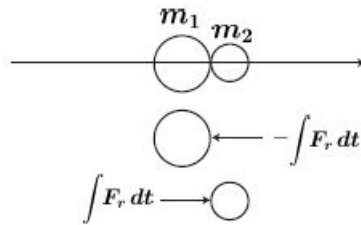$$-\int F_d \, dt \;=\; m_1 v_G - m_1 v_1 \qquad \text{(deformation phase)}$$

and for particle 2

$$\int F_d \, dt \;=\; m_2 v_G - m_2 v_2 \qquad \text{(deformation phase)}$$

IV) Restoration

The contact force F_r decreases and the particles move apart.

The impulse applied to particle 1 and 2 from the restoring force equals the change in momentum in the restoration process. After the restoration process the velocity of m_1 is v_1' and the velocity of m_2 is v_2'. For particle 1
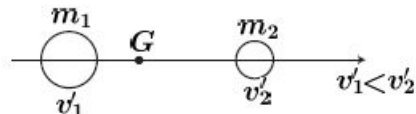
$$-\int F_r \, dt \;=\; m_1 v_1' - m_1 v_G \qquad \text{(restoration phase)}$$

and for particle 2

$$\int F_r \, dt \;=\; m_2 v_2' - m_2 v_G \qquad \text{(restoration phase)}$$

V) After Impact

The particles travel with a constant velocity v_1' and v_2'.

From momentum conservation, the total momentum before and after impact should remain the same, i.e. $m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$, and therefore the velocity of the center of mass will remain unchanged $v_G' = v_G$, thus $m v_G = m_1 v_1' + m_2 v_2'$. If $g' = v_1' - v_2'$ is the relative velocity of the two particles after impact, then

$$v_1' = v_G + \frac{m_2}{m} g' \tag{5}$$

$$v_2' = v_G - \frac{m_1}{m} g' . \tag{6}$$

Coefficient of Restitution

We define the coefficient of restitution as the ratio between the restoration and deformation impulses.

$$e = \frac{\int F_r \, dt}{\int F_d \, dt}$$

For physically acceptable collisions $0 < e < 1$. The value of $e = 1$ corresponds to an *elastic* collision, whereas the value of $e = 0$ corresponds to a totally *inelastic* collision in which the restoration impulse is equal to zero.

We can consider each particle separately and set the impulse on the particle equal to the change of linear momentum

Since for particle 1

$$-\int F_d \, dt = m_1 v_G - m_1 v_1 \qquad \text{(deformation phase)}$$

$$-\int F_r \, dt = m_1 v_1' - m_1 v_G \qquad \text{(restoration phase)}$$

Thus, $e = (v_1' - v_G)/(v_G - v_1)$, or $(e+1) v_G = v_1' + e v_1$.

and for particle 2

$$\int F_d \, dt = m_2 v_G - m_2 v_2 \qquad \text{(deformation phase)}$$

$$\int F_r \, dt = m_2 v_2' - m_2 v_G \qquad \text{(restoration phase)}$$

Thus, $e = (v_2' - v_G)/(v_G - v_2)$, or $(e+1) v_G = v_2' + e v_2$.

Combining the above expressions, we have

$$e = \frac{\int F_r \, dt}{\int F_d \, dt} = -\frac{v_1' - v_2'}{v_1 - v_2} = -\frac{g'}{g} \ ,$$

which expresses the coefficient of restitution as the ratio of the relative velocities after and before the collision. Finally, if we know e we can use equations (5) and (6) to obtain

$$v_1' = v_G - \frac{m_2}{m}eg$$
$$v_2' = v_G + \frac{m_1}{m}eg \ .$$

Example   **Simple Collisions**

As an example, we consider a simple case of a mass m_2 at rest and a mass of m_1 traveling at velocity v_1 headed for a collision with m_2. After the collision, the velocities of the two masses will in general be non-aero. We consider the cases $e = 1, .5, and 0$, the is a range of coefficient of restitution between 1, perfectly elastic, to zero, perfectly inelastic. The result for the final velocities of m_1 and m_2, normalized by the initial velocity v_1, are given by the equations above and shown in the figure.
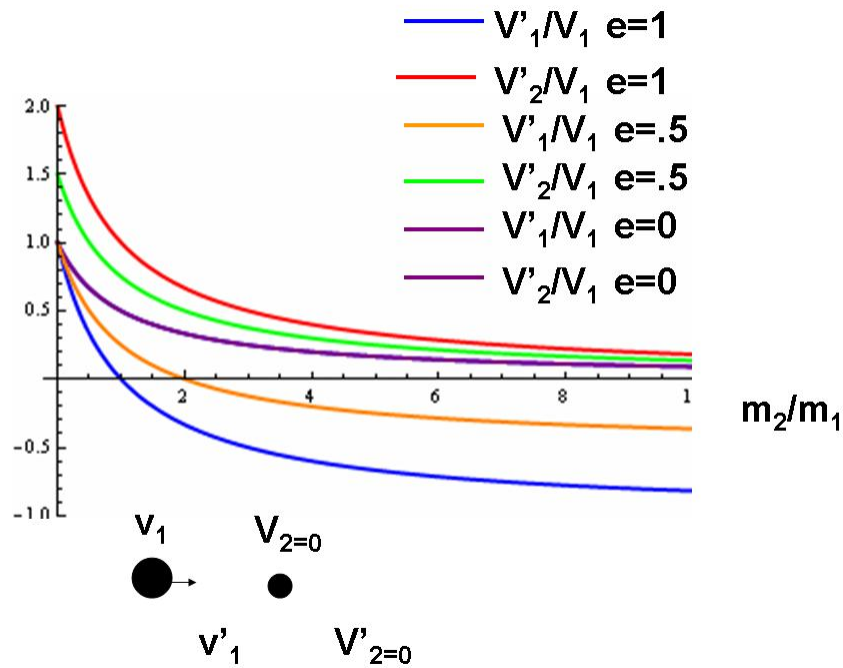
For $e = 0$, these is only one curve since the particles stick together: $v_1'/v_1 = v_2'/v_1$. For the elastic collision, note the result for equal masses: $v_1' = 0$ and $v_2'/v_1 = 1$

Energy

We can now look at the kinetic energy before and after the impact. The kinetic energy before impact is,

$$
\begin{aligned}
T &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \\
&= \frac{1}{2}m_1\left(v_G^2 + 2\frac{m_2}{m}g v_G + \frac{m_2^2}{m^2}g^2\right) + \frac{1}{2}m_2\left(v_G^2 - 2\frac{m_1}{m}g v_G + \frac{m_1^2}{m^2}g^2\right) \\
&= \frac{1}{2}m v_G^2 + \frac{1}{2}\frac{m_1 m_2}{m}(v_1 - v_2)^2 = \frac{1}{2}m v_G^2 + \frac{1}{2}\mu g^2.
\end{aligned}
$$

Here we introduce the notation $\mu = (m_1 m_2)/m$; this is often referred to as the reduced mass of the pair. The reduced mass approaches $1/2$ of each mass when the two masses are similar, but it approaches the lighter mass if the masses are dissimilar. The kinetic energy after impact will be

$$
\begin{aligned}
T' &= \frac{1}{2}m v_G^2 + \frac{1}{2}\frac{m_1 m_2}{m}(v_1' - v_2')^2 \\
&= \frac{1}{2}m v_G^2 + \frac{1}{2}\frac{m_1 m_2}{m}g'^2 \\
&= \frac{1}{2}m v_G^2 + \frac{1}{2}\mu e^2 g^2.
\end{aligned}
$$

since $g' = eg$ and $\frac{m_1 m_2)}{m} = \mu$. Therefore, the kinetic energy lost during the collision is

$$
\Delta E = T - T' = \frac{1}{2}\mu(1 - e^2)g^2,
$$

which is zero for $e = 1$ (elastic collision), and maximum when $e = 0$.

Momentum exchange

We can also look at the momentum which is exchanged between particle 1 and particle 2,

$$
M_{12} = m_1 v_1 - m_1 v_1' = m_1\left(v_G + \frac{m_2}{m}g - v_G + \frac{m_2}{m}eg\right) = (1 + e)\mu g.
$$

This, is maximum when $e = 1$ (elastic collision) and minimum when $e = 0$.

It is particularly illuminating to examine a special case of collisions, that for which the two masses are equal.

Equal masses
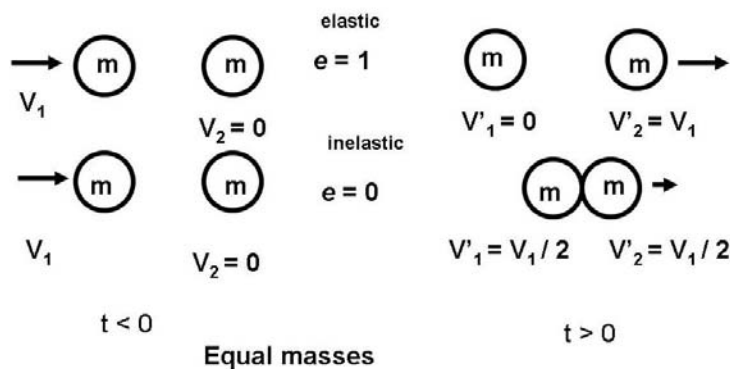
We consider two cases shown in the figure. In both cases one particle is at rest, the other approaching with velocity v_1. In one case, we assume an elastic collision, $e = 1$; in the other a completely inelastic collision,

$e = 0$. In the first case, kinetic energy is conserved so it is obvious–although you can work out the algebra–that the first particle completely transfers its momentum to the second particle, where upon it continues to the right with the original velocity $v_2' = v_1$. In the second case of a completely inelastic collision, there is no rebounding force and the particles stick and move together. Conservation of momentum requires the $v_2' = v_1/2$. Kinetic energy is not conserved.

***Example***   ***Bouncing Ball***

If a ball, initially at rest, is released on a flat surface from a height h and it rebounds to a height h', the coefficient of restitution is given by

$$e = -\frac{v'}{v} = \sqrt{\frac{h'}{h}},$$

where v and v' are the velocities before and after impact.

Oblique Collisions

In the case of oblique collisions, we consider the instant of impact and define the normal direction, n, along the line that connects the two mass centers, and the tangential direction, t, along the line tangent to the surfaces at the point of contact.
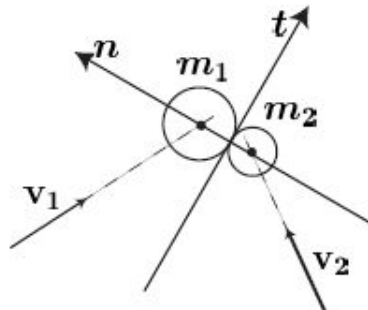
We write the equations of conservation of momentum in the tangent and normal directions. Since the force of contact is assumed to act along the normal direction, the conservation of linear momentum along the tangential direction implies

$$m_1(v_1)_t = m_1(v_1')_t \quad \Rightarrow \quad (v_1')_t = (v_1)_t$$
$$m_2(v_2)_t = m_2(v_2')_t \quad \Rightarrow \quad (v_2')_t = (v_2)_t$$

In the normal direction, we solve a 1D collision problem, that is,

$$m_1(v_1)_n + m_2(v_2)_n = m_1(v_1')_n + m_2(v_2')_n$$
$$e = -\frac{(v_1')_n - (v_2')_n}{(v_1)_n - (v_2)_n}$$

which determine $(v_1')_n$ and $(v_2')_n$.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

3/8, 3/9

16.07 Dynamics
Fall 2009