

# Chapter 2: Deriving AdS/CFT

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## Lecture 17

Important equations for this lecture from the previous ones:

1. The global  $AdS_{d+1}$  can be described as a hyperboloid in a flat Lorentzian spacetime of signature  $(2, d)$ , equation (13) in lecture 16:

$$X_{-1}^2 + X_0^2 - \vec{X}^2 = R^2 \quad (1)$$

2. The Poincare patch, equation (12) in lecture 16:

$$ds^2 = \frac{R^2}{z^2} \left( -dt^2 + dz^2 + d\vec{x}^2 \right) \quad (2)$$

3. The relation between the gravitational constant  $G_N$  and string theory's  $g_s$  and  $\alpha'$ , equation (15) in lecture 12:

$$G_N = 8\pi^6 g_s^2 \alpha'^2 \quad (3)$$

### 2.3.1: AdS SPACETIME (cont.)

Since the global  $AdS_{d+1}$  can be defined as in equation (1), it is clear that the spacetime has an  $SO(2, d)$  isometry, which is the same as the conformal group in flat  $d$ -dimensional Minkowski space. This can be seen from the Poincare patch, which is given in equation (2):

1.  $P^\mu$ :  $d$  generators, translations along  $x^\mu = (t, \vec{x})$ .
2.  $M^{\mu 0}$ :  $\frac{1}{2}d(d-1)$  generators, Lorentz transformation for  $x^\mu$ .
3. Scaling: 1 generator, take  $z$  to  $\lambda z$  and  $x^\mu$  to  $\lambda x^\mu$ .
4. Special conformal transformation (SCT):  $d$  generators

$$z \rightarrow z' = \frac{z}{1 + 2bx + b^2A} \quad , \quad x^\mu \rightarrow x'^\mu = \frac{x^\mu + b^\mu A}{1 + 2bx + b^2A} \quad , \quad A = z^2 + x^2 \quad (4)$$

One can try to understand the SCT as the combination of inversion  $z \rightarrow \frac{z}{A}$  and  $x^\mu \rightarrow \frac{x^\mu}{A}$ , translation by  $b$ , then the very same inversion again.

Altogether, there are  $\frac{1}{2}(d+1)(d+2)$  generators.

### 2.3.2: STRING THEORY IN $AdS_5 \times S^5$

A perturbative string theory can be characterized by  $g_s$  and  $\alpha'$ . Since  $AdS_5 \times S^5$  is a homogeneous spacetime of constant curvature everywhere, string theory in this spacetime background has only 2 dimensional parameters,  $g_s$  and  $\frac{\alpha'}{R^2}$ . As  $G_N$  is related to  $g_s$ , given in equation (3), another choice for parameters (which is more convenient in the classical gravity limit) is with  $\frac{G_N}{R^8}$  and  $\frac{\alpha'}{R^2}$ .

The classical gravity limit gives IIB SUGRA:

$$g_s \rightarrow 0 \quad , \quad \frac{\alpha'}{R} \rightarrow 0 \quad , \quad \frac{G_N}{R^8} \rightarrow 0 \quad (5)$$

The classical string limit:

$$g_s \rightarrow 0 \quad , \quad \frac{\alpha'}{R^2} \rightarrow \text{finite} \quad (6)$$

The compactness of  $S^5$  make it convenient to express a 10D field in terms of a tower of  $AdS_5$  fields by expanding it in terms of 5-spherical harmonic function, for example, with a scalar:

$$\Phi(x^\mu, z, \Omega) = \sum_l \Phi_l(x^\mu, z) Y_l(\Omega) \quad (7)$$

As the gravity is essentially 5D, the graviton zero-mode on  $S^5$  with  $V_5$  is the volume of  $S^5$ :

$$\frac{1}{16\pi G_N} \int d^5x d^5\Omega \sqrt{-G_{10}} \mathcal{R}_{10} = \frac{V_5}{16\pi G_N} \int d^5x \sqrt{-G_5} \mathcal{R}_5 \quad (8)$$

Hence, the effective 5D gravitational constant:

$$G_{N5} = \frac{G_N}{V_5} = \frac{G_N}{\pi^3 R^5} \quad (9)$$

After dimensional reduction on  $S^5$ , then the action can be written as:

$$S = \frac{1}{16\pi G_{N5}} \int d^5x \left( \mathcal{L}_{gravity} + \mathcal{L}_{matter} \right) \quad (10)$$

### 2.3.3: $\mathcal{N} = 4$ SYM theory

The fields content of interests are  $A_\mu$ ,  $\Phi^i$  (with  $i = 1, 2, \dots, 6$ ) and the Weyl spinor  $\chi_\alpha^A$  ( $A = 1, 2, 3, 4$ ), they are all in the adjoint representation of  $U(N)$

. Altogether, the number of on-shell degrees of freedom:

$$\left( 8(\text{bosonic}) + 8(\text{fermionic}) \right) \times N^2 = 16N^2 \quad (11)$$

The interacting part are actually  $SU(N)$  while the  $U(1)$  part decouples and free:

$$\Phi^i = \Phi_a^i T_{SU(N)}^a + \phi^i T_{U(1)} \quad , \quad A_\mu = A_\mu^a T_{SU(N)}^a + B_\mu T_{U(1)} \quad ; \quad T_{U(1)} \sim 1_{N \times N} \quad , \quad \left[ T_{U(1)}, T_{SU(N)}^a \right] = 0 \quad (12)$$

Specifically, the  $\phi^i$  and  $B_\mu$  fields (decouple from  $SU(N)$  degrees of freedom) governed by a free theory.

There's a subtlety in why the AdS bulk space is equivalent to  $SU(N)$  but not the whole  $U(N)$ . Indeed,  $U(1)$  actually corresponds to the translational degree of freedom of the whole  $N$  D3-branes, so by taking that group on one side and the D-branes out of the other, one left with the correspondence between the CFT  $\mathcal{N} = 4$   $SU(N)$  SYM and IIB stringy physics in  $AdS_5 \times S^5$  background. And there are many other ways to see that, such as in the dual string theory in  $AdS_5 \times S^5$  all modes couple to gravity, therefore the noninteracting part  $U(1)$  shouldn't be included.

Also, note that at tree level, correlation functions of  $U(N)$  and  $SU(N)$  are the same, but in general it's not true at higher loop orders. However, as  $N$  goes to infinity, then even at loop orders these 2 gauge groups are in agreement.

The piece of string theory of interests is about the center of mass motion of the  $N$  D3-branes:

$$\mathcal{L} = -\frac{1}{g_{YM}^2} \text{Tr} \left( \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \left( D^\mu \Phi^i \right) \left( D_\mu \Phi_i \right) + \left[ \Phi^i, \Phi^j \right]^2 \right) + \text{fermionic} \quad (13)$$

The properties of the theory:

1. It has  $\mathcal{N} = 4$  SUSY: in (1+3)D, with  $\mathcal{N} = 1$  the supercharge  $Q_\alpha$  is a Weyl spinor, while  $\mathcal{N} = 4$  has 4 of such conserved  $Q_\alpha^A$  supercharge ( $A = 1, 2, 3, 4$ ), maximally allowed SUSY for a renormalizable field theory.
2. The SYM coupling  $g_{YM}$  is dimensionless classically, and in the quantum theory one also get the  $\beta$ -function of the coupling to be zero, therefore it is a genuine dimensionless parameter.
3. The theory is conformally invariant, which is a conformal field theory (CFT), because the vanishing of  $\beta$ -function indicates that the theory doesn't have a scale.

The full bosonic symmetries are  $SO(2, d) \otimes SO(6)$ , and by including SUSY, one arrives at the superconformal symmetry (SCFT)  $PSU(2, 2|4)$ . Since  $\mathcal{N} = 4$  is the most symmetry (thus likely simplest) among all interacting theories in (1+3)D.

In a CFT, the basic objects are local operators with a definite scaling dimension  $\Delta$  (conformal dimension):

$$O(x) \rightarrow O'(x) = \lambda^\Delta O(\lambda x) \quad (14)$$

The typical observables are correlation functions of local operators. The conformal symmetries determine 2-point and 3-point correlation functions up to a constant:

$$\langle O(x_1) O(x_2) \rangle = \frac{C \delta_{\Delta_1, \Delta_2}}{|x_1 - x_2|^{2\Delta_1}} \quad , \quad \langle O(x_1) O(x_2) O(x_3) \rangle = \frac{C_{123}}{|x_1 - x_2|^{\Delta_1 + \Delta_2 - \Delta_3} |x_2 - x_3|^{\Delta_2 + \Delta_3 - \Delta_1} |x_3 - x_1|^{\Delta_3 + \Delta_1 - \Delta_2}} \quad (15)$$

Some comment about the equivalent between  $\mathcal{N} = 4$  SYM with gauge group  $SU(N)$  in  $\mathbb{R}^{1,3}$  and type IIB string in  $AdS_5 \times S^5$  (Poincare patch):

1.  $\mathcal{R}^{1,3}$  is the boundary of the Poincare patch of  $AdS_5$
2. IIB string theory in such background gives a 5D gravity theory.

The equivalence can be considered as a realization of the holographic principle, and this lead to a nontrivial prediction that  $\mathcal{N} = 4$  on  $S^3 \times \mathbb{R}$  is equivalent to type IIB string theory in the global  $AdS_5 \times S^5$ .

Why we can (and certainly should) identify the CFT  $\mathcal{N} = 4$   $SU(N)$  SYM on Minkowski space of (1+3)D (D3-branes worldvolume) with the boundary of  $AdS$ ? The reason is that it's the only possible  $SO(2, 4)$ -invariant between these 2 spaces. Also, the AdS/CFT mapping is naturally formulated with bulk/boundary [1] (indeed, this can be guessed from that the AdS boundary condition should be controlled by the CFT of interests). Another evidence is that the number of degrees of freedom seems to be matched. Hence, it is expected that the relation is holographic.

For a quantum gravitational theory, spacetime fluctuates, so what do we really mean by  $AdS_5 \times S^5$ ? Well, indeed, for finite  $g_s$ , the quantum gravitational fluctuation can be large, one should interpret  $AdS_5 \times S^5$  as specifying the asymptotic structure of the bulk spacetime (the boundary conditions for the bulk quantum gravity).

## Reference

- [1] Witten, E. (1998). *Anti de Sitter space and holography*. arXiv preprint hep-th/9802150

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