HONG LIU: OK, let us start. Welcome, everybody. So first, let me just say a few general things regarding the logistics of this class.

So you should have these three handouts. One, organization. One, outline. And then, one is for PSET 1. So let me just go over briefly this organization and the outline.

So firstly in this class, there's no textbooks. However, there are many reviews available. But unfortunately, a lot of those reviews, none of the single review is suitable for the whole course.

Yeah. So I have listed some of the reviews at the end of this organization. Yeah. So along the way, I will point it out, some specific references or some specific parts of certain reviews if they are directly relevant.

And also for this class, there's no recitation. Yeah, for advanced courses like this, department does not assign TA. So we don't have recitation. So I have my own office hour, which is Monday 3:30 to 4:30. Just one hour before the lecture on Monday. Yeah. So if you have questions or if you have-- yeah, just feel free to come.

So if you have some questions other time, say, if this office hour is not good for you, I'm also happy to set up some other time to meet you. Or, you can stay after the class to ask questions, et cetera. Yeah. So whichever way is more convenient to you.

So today, we have this-- yeah, so form now on, everything will be just on the web. You should download the future PSETs. Or if I put any lecture notes or materials, you should find them on the web. And I assume you can all find out the website of the course. And it's also listed here in the organization page. In the organization page.

So any questions so far? Good.

So this class there's no exam. And the grades are solely based on PSETs and the final project. And the PSETs will be due every two weeks.

So if you look at the calendar, there are about like 5 PSETs. Then, for the last three weeks, instead of, say, handing a PSET, you're handing in the small paper. Just whatever topics you would like to choose about the holographic duality. Yeah, just choose one topic. Write, say, 8 to 10-page paper. Either review or some calculation you would like to do, et cetera. Yeah.

So 5 PSET is 75%. And this final project would be 25%. That's pretty much the logistics regarding this class. Do you have any questions regarding the grading, regarding the project or PSET, et cetera? Good. So let me say a few words about this outline.

So this outline is way too ambitious. It gives you-- yeah, it just serves to give you a rough contour what we will cover. But really, don't take it too literally because-- first, we are not going to cover all seven chapters. That would be too much.

So last time when I taught it, I covered only three chapters. And so hopefully, this time I will do four. Hopefully, time I will do four. And I may also just deviate from some of the things here, depending on the pace or depend on people's interests, et cetera. So this is going to be flexible.

If there are certain things you would really very much like to hear, you can also let me know. I can think about it. In particular, among the five, six, seven. And I can think about it, maybe I'll discuss some of them. Discuss some of them.

Yeah. Also, another thing if you keep in mind is that in this class, this is a class which will touch on many different subjects, say gravity, a little bit of string theory. And also, quantum field theories, et cetera. So in such a class, it's almost impossible-- I think it's just impossible to really derive everything in the self-contained way. And also, some of-- yeah, I think some of your background are also very different. So the certain part may not appear self-contained to some of you.

So if there is anything which is not familiar to you, if I just mention it, if I just code it, if it's not familiar with you, then you make sure-- don't hesitate to ask me, either during the class or after the class, so that I can help you to make up on that. To make up on that.

And also, to keep in mind, like other advanced graduate classes, you actually mostly learn outside the class. Inside the class, it's mostly try to provide you some motivations, guidance, et cetera, and give you a rough contour. And your own reading outside the class actually should be the main routes you learn things. The main route you learn things. So any questions?

So during the class, please feel free to ask questions. Yeah. Asking questions are a great thing because if something you don't understand, there's a very good chance quite a few of your fellow students also don't understand.

So if you ask the question, you're not only helping yourself, you're also helping other people. And you also help me because then I will know certain aspect is not clear to you. Then, I will try to emphasize it, or repeat it, et cetera. So please, do ask questions.

And also, if you have any feedback on the class, whether it's too fast, too slow, or problems are too hard or too easy, the PSET is too short for you, just not challenging enough, just let me know. Just let me know. Yeah. Then, I will try to adjust.

And again, this is a very important. Again, this is very important because the-- yeah. This will help us to improve the class. To improve the class. Any other questions? Or any questions? OK. Then, let us start our lecture.

So first, we look at the hints for the holographic duality. OK. So we start by doing a multiple choice problem.

OK. So what is your answer?

AUDIENCE: [INAUDIBLE]

HONG LIU: Good. So why is gravity?

AUDIENCE: We don't have a good quantum theory of gravity.

HONG LIU: Good. So we don't have a good quantum theory of gravity. Any other reason?

AUDIENCE: Except string theory.

HONG LIU: Yeah, string theory is still not the-- I would say not yet at the [INAUDIBLE] quantum theory of gravity.

AUDIENCE: [INAUDIBLE]

HONG LIU: Yeah. Yeah, this is the similar to that. We don't have a full theory of gravity nonrealizable.

AUDIENCE: [INAUDIBLE]

HONG LIU: Good. Anything else? Or any other opinions? Say you should choose any other interaction other than gravity?

AUDIENCE: Strong is the strongest one. Strong is the strongest one.

HONG LIU: That's right. You can also choose the gravity because this is the weakest one. Yeah, so the gravity is the weakest one. OK. So yeah. So I think you have said most of the things. And the gravity is very different from the others. And in some sense, all the others are understood. So a to c are now understood to be described by gauge theories in the fixed spacetime. So the fixed spacetime. In fact, mostly we use Minkowski. Minkowski.

And for example, electromagnetism, [INAUDIBLE] by QED. And the QED [INAUDIBLE] gauge field. You want gauge theory plus a Dirac fermion. Dirac theory. A Dirac theory.

And similarly, if you include the weak interaction-- so there's something called [INAUDIBLE] weak-- it's described by su2 times u1 gauge theory. Described by su2 times u1 gauge theory.

And also, the strong interaction you know now is described by su3 gauge theory. So for all those cases, the basic theoretical structure is understood. So the basic. OK.

In principle, we can formulate those theories from first principle, just use path integral. Say, [INAUDIBLE] where [INAUDIBLE] group just based on principle, relation group. And then you can actually formulate all the theory in the [INAUDIBLE] way. So in the sense, then any calculation of theories can be reduced to certain algorithm. OK.

Of course, this does not imply we actually know how to calculate many quantities here. It just say in principle, they are calculable. We often don't have the technical tool to calculate them. For example, even in QCD. [INAUDIBLE] joint action, we don't have the technical tool to calculate certain things. But they are, in principle, calculable. And we can write down the algorithm to calculate it.

But for gravity, this is different. So for gravity, it's very different.

So for gravity. So now we understand that classical gravity, it's just equal to spacetime. It's just a theory of spacetime. So this is the content over the general relativity. The general quantum activity.

But at a quantum level, we really don't have a precise idea how to formulate the theory. So there are many questions still not understood. Many, say, conceptual questions still not understood. For example, a spacetime should fluctuate. OK. The spacetime should fluctuate because of the quantum fluctuation [INAUDIBLE] in quantum anything fluctuates.

And then, the natural question is whether this is still-- so natural question is whether spacetime is still continuous. OK. So this is still [INAUDIBLE]. You need to replace it by something discrete.

And it's not even clear right now whether spacetime will remain a fundamental notion when you go to quantum gravity. It could well be that spacetime would be replaced by something else. And also, there are many other questions.

So for example, quantum nature of black holes. Or what should be the beginning of the universe, et cetera. So all these questions are not understood.

Also, gravity is the weakest interaction. It is actually much, much weaker. We will see, it's much, much weaker than all the other interactions. And also, people have been speculate-- maybe actually this feature can also be actually a fundamental feature underlying in the special role of the gravity.

For many years, it seemed to people that these questions and the gravity questions, they're completely unrelated. They're completely unrelated. They're a completely different subject. And those are well understood, but somehow those we need completely new ideas. So it came out a great surprise. So it came out that great surprise in 1997 by [INAUDIBLE] that actually they actually are equivalent. So the proposal of the [INAUDIBLE] is that the-- chalk. Is that quantum gravity is actually equal to field theories on a fixed spacetime.

So on a fixed spacetime just means that there's no gravity here on the right-hand side. Because if there's gravity, then you cannot have a fixed spacetime. And spacetime should be dynamical. So this is the equivalence between quantum gravity and just our ordinary quantum field theories. And this is ordinary quantum mechanics. And this is a very nontrivial quantum mechanics, which we don't yet understand. OK.

This equality should be really understood as the different descriptions of the same quantum systems. So two sides are different descriptions of the same quantum system. So it just depends on how you want to see it. So one way you see it, you see a quantum gravity. Then from some other way you see it, you don't see gravity at all. You just see ordinary quantum mechanics. OK.

Yeah, so this is what this equality is supposed to mean. So this equality essentially-- let me call this equation 1, which I will often use later. So this equation 1 is really a unification. Unification, which unifies quantum gravity with other field theory in a rather unconventional way. Not in a standard sense of unification. But that's bringing them together. OK.

So now, let me make some philosophical remark. Now, let me make some philosophical remark. So last time, I have checked when preparing this class this [INAUDIBLE] original paper has been cited more than 10,000 times in the [INAUDIBLE] Database. And so there's a huge amount of work has been devoted to understanding this relation.

But in my opinion, this subject's still in its infancy. And with many elementary [INAUDIBLE] we will see in the course as course goes on, there's still many elementary issues which are not understood. And to us, this relation is still like a magic. And we don't really have very good idea where this relation comes home. We have some rough idea, but not very precise idea.

So the purpose of physics, of course, is to turn what looks like a magic or art into some rules so that it becomes something trivial. So at the beginning of course of the gravity, Newtonian gravity was magic. But in the end-- and what Newton did is to make the gravity to be not different from the force we see in ordinary life. And similarly, for other-- yeah, for other interactions we have seen before. Anyway.

So personally, I believe when we really fully understand this relation, this will really be a huge landmark in the physics. And comparable, say, maybe to Newton's understanding of gravity or Maxwell-Boltzmann, et cetera. So the goal of this course is to help you understand how we-- our current understanding of this relation. And also, to help you to derive the related-- yeah, so these are two very different objects.

And in order to set up an inequality, you also need to define a dictionary. For example, this is like Chinese. This is like English. So you need to define a dictionary between the two and what one said to the other. So we will also work out the dictionary. And also, we will develop with the tools and how to use this relation. And how to use the relation. And also, discuss many features of this relation and the implications.

So the purpose is to help you to-- really, to bring to the forefront of this very exciting subject. So any questions? Good. No questions? OK, so now let's move to-- so after this introduction. So let's talk about this. OK.

So if you look at this relation. So looking from the right-hand side. So you just have some kind of ordinary field theory-- ordinary quantum mechanical system, which we, in principle, know very well conceptually, without any gravity at all. Somehow, if you view it in some different way than the gravity and the spacetime, and the dynamical spacetime should emerge out of this.

So in some sense, this 1 implies that the quantum gravity-- so let me just use the quantum gravity plus the dynamical spacetime. OK, dynamical spacetime. So it's important it's a dynamical spacetime-- can really merge from non-gravitational system. It can emerge from a non-gravitational system.

So the idea of emergence of gravity from some other degrees freedom is not a really new idea. So [INAUDIBLE] in 1967, Sakharov, who was a Russian physicist. Sakharov is a very common name, a very common Russian name. But this is the most famous one. This is the most famous Sakharov who has invented the hydrogen bomb for the former Soviet Union. And he was also the guy who also later, I think, got Nobel Peace Prize for some human rights stuff. But he was also a very excellent physicist, including this idea of emergence of the gravity.

So what he observed is that-- he found actually certain [INAUDIBLE]. He would study, say, some materials, et cetera. Then, he finds that certain [INAUDIBLE] systems can actually have mathematical description which looks like a magic connection, et cetera. Yeah, just mathematical, it looks like the equation for the magic and the equation for the connection, et cetera.

And then he was speculating, maybe the general relativity can actually arise just from some kind of electron systems, just from the ordinary [INAUDIBLE] systems, and as effective description.

And in fact, in the 1950s, even before the Sakharov have [INAUDIBLE] idea, people working in GR, in General Relativity, they already found many striking parallel between Einstein's equations and also associated many phenomena we hydrodynamics. And so we know that hydrodynamics is really just an effective description.

And if you just look at the river-- so in fact, it's the discrete water molecules. But if you describe the motion of the water molecules at a microscopic level, then you find the hydrodynamics. But the hydrodynamics does not apply at more fundamental levels. So it's more like some kind of effective theory.

So in the 1950s, people already found that there are many features of Einstein equation-- it's reminiscent. Actually, just reminiscent of certain features of hydrodynamics. So there's already this suspicion that maybe the gravity or spacetime actually can emerge out of something else. Just like the hydrodynamics is not a fundamental theory. It actually emerges from molecules, dynamics of the molecules. Anyway.

So from the field theory perspective, it's also natural to ask whether massless spin-2 particles can arise as bound state, say, of lower spin particles. Say, like spin-1. Like photons, which are spin-1. So photons. Or gluons in QCD. Electrons, or quarks, et cetera. So these are the spin [INAUDIBLE] objects.

And if yes-- so massless spin-2 particle in some sense-- so when you learn QR, you might have learned that the massless spin-2 particle may be considered as a hallmark form of gravity because of the-- so the basic propogation. The gravitational wave. So the basic propogation of the gravity is the gravitational wave. And that's propogated as spin-2 particles.

And if you quantize the theory-- a propogation as spin-2 object. And when you quantize the gravitational waves, then they become massless spin-2 particles. They become spin-2 particles. So in some sense, if your theory can allow the massless spin-2 particles as-- so if your theory contains the massless spin-2 particles, then they must contain gravity. OK.

So for example, even in strong interaction, which are, say, the theory describes gluons and the quarks. And the gluons and the quarks, they can indeed form. So these are the gluons and the quarks.

In [INAUDIBLE], they're indeed spin-2 bound states. The gluons are spin-1, quarks are spin [INAUDIBLE]. But you combine them together, you can make [INAUDIBLE] the spin-2 bound state, even just in strong interactions.

But of course in nature, the spin-2 bound state, they're all massive. They're all massive. And there are some very unstable, massive particles. When you create them, immediate decay. So they cannot be gravity.

For gravity, you need massless spin-2 particles. But if you look at this fact, you cannot have the feeling. Say, maybe I take the QCD. Maybe I take a little bit. Maybe I can make that spin-2 massive particles into some massless spin-2 particles. Then I would have generated the gravity from a QCD like series. And then, that would be a revolution. And then you will be immortal.

But unfortunately, even though this was a very promising idea for many years, and this hope was actually completely dashed by a powerful theorem of Weinberg and Witten. So there's a powerful theorem from Weinberg-Witten that say this can never happen. So this is the-- they're not possible. So this is 1980.

So the paper. If you want, you can take a look at the paper. So this is volume 96 and the page 59. So they proved two theorems in that paper. So let me just copy the theorem down. Copy the theorem down.

So they say a theory that allows the construction of a Lorentz-covariant and the conserved current for a vector-- current. Say, j mu, cannot contain massless particles. Cannot contain massless particles of spin greater than 1/2 with non-vanishing value of-- OK, so this is first theorem.

So the second theorem-- I think I should have enough space. So the second theorem. Second theorem says that a theory that allows a covariant, Lorentz-covariant, and the conserved stress tensor. So let me just call it T mu mu. A stress tensor cannot contain massless particles of spin j greater than 1.

So the key-- so let me just repeat the theorem a little bit. So the first theorem said if you have a Lorentz-covariant and the conserved current, and in such a theory there's no charged particles can have spin more than 1/2. So there's no charged particle can have spin more than 1/2. Of course, this cannot contain graviton.

And the second theorem says if you have a conserved stress tensor, Lorentz-covariant and a conserved stress tensor, then this theory cannot contain any particles with spin greater than 1. So of course, this cannot also contain a graviton. Graviton would be a spin-2.

So this theorem-- yes?

AUDIENCE: Is there anything like RG flow of mass? [INAUDIBLE]

HONG LIU: This is final mass. Yeah. Just the [INAUDIBLE] mass. You don't talk about even normalization here. Yeah, this is just [INAUDIBLE] statement.

So this theorem turned out to be actually very easy to prove. And in some sense, it's rather instructive. And just to give you the sense of the power of this theorem, we will prove it in a little bit. So before proving it, I will first make some remarks to make you appreciate what these two theorems means. Do you have any questions before I do that? OK, good. So let me just make some remarks on this theorem.

So the first remark is that the theorems really apply to any particles, to both. So normally we say the fundamental particle is the particle which you put in your theory is the particle which say-- yeah, let me say [INAUDIBLE]. Rather fundamental, let me call it elementary. So the particle which already appear in your Lagrangian. And composite particles. Composite particles, say, would be some kind of bound state or some particles which don't appear in your original Lagrangian. So this theorem does not distinguish them.

As far as you have some particles, [INAUDIBLE] region. So now, let's try to see whether this theorem is compatible to [INAUDIBLE] these things we already know. So let's first try to apply to, say, QED.

So the theorem is compatible with QED. So we know that QED is a Maxwell theory plus Dirac's theory. And then the Dirac-- and the fermions in the Dirac theory interact with the photons in the Maxwell theory. So this theory have a massless photon of spin-1. And this will have a conserved charge because of, say, electrons that do have a charge. This theory has a conserved current. Has a conserved current.

But this theory does allow a spin-1 particle. It does allow the spin-1 particle. And this is not contradictory with the theorem 1. So theorem 1 said if you have a Lorentz [INAUDIBLE] and the conserved current, then you cannot have massless particle of spin 1/2 with non-vanishing charge.

In this case, photon is massless. But photon does not carry charge. Photon is neutral under the electric charge. So the existence of the photon is compatible with that theorem.

And now, let's look at the [INAUDIBLE] theory which also contain massless spin-1 particle. And the [INAUDIBLE] theory which contain the spin-1 particle is the Yang-Mill theory. So Yang-Mill theory. For example, let's consider su2 Yang-Mill theory. So the Yang-Mill theory have three gauge field, A mu a. And a is equal to 1, 2, 3. Because 1 and 2 [INAUDIBLE] of su2 gauge symmetry.

And from this three gauge field, you can combine them into the following form. I have A mu 3. Then, I have A mu plus minus. You could do 1 over square root 2 a mu 1 plus minus A mu 2. OK. So [INAUDIBLE]. So you can easily check yourself that this a mu plus minus is massless spin-1 particle. Particles. Can create massless spin-1 particles charged under the u1 subgroup generated by sigma 3.

So sigma 3 is the generator associated with A mu 3. And so those two are charged under this generator. So now, [INAUDIBLE] we have a contradiction because those are massless spin-1 particles charged under this. And the theorem 1 said you have a conserved current, then you cannot have any particle recharged-- any charged particle with spin greater than 1/2. But those are the spin-1 particles.

But actually, we're consistent. Because as you may remember, in Yang-Mills theory there does not exist-- that actually does not exist. The way out [INAUDIBLE] there does not exist a conserved Lorentz-covariant current for actually-- for this u1. For the u1 generated by this sigma 3 divided by 2. So this is actually compatible. This is actually compatible because such a current does not exist. OK.

So you will show this fact yourself in your PSET. But you may remember, if you have studied quantum field theory 2, then you may remember this fact from the discussion of [INAUDIBLE] gauge theories. So in this course, we will from time to time use various facts of, say, [INAUDIBLE] theories. But they're not essential. If you don't allow [INAUDIBLE] theories, you can still understand this duality. You can still understand the relation one. But of course, if you know gauge field, it will help you a lot.

Any questions about this? Good. Let's continue. So this theorem does not forbid graviton from GR. So if you try to quantize Einstein's general relativity around the flash space for example, then you will find the massless spin-2 particle. So this is what we call graviton. And this is not contradictory with that theorem 2. It's because you may still remember when you learn GR is that in GR, General Relativity, actually there is not. There's no conserved. So here, it's conserved. Not covariantly converged, just conserved. Lorentz-covariant stress tensor. OK.

So in GR, actually there's no conserved in the Lorentz-covariant stress tensor. So actually, the GR [INAUDIBLE] graviton.

But this theorem 2, it's nevertheless, very powerful. It's nevertheless very powerful. It said very powerful. It said all [INAUDIBLE]. None of [INAUDIBLE] QFTs in Minkowski spacetime, which is the one we normally work with, can have emergent gravity. OK. So QCD like theories, no matter how you try to trick the theory, the spin-2 massive bound state can never become massless. They can never become massless. Because those series, they do-- for example, our QCD, our strong interaction, they do have a covariantly conserved stress tensor. A Lorentz-covariant conserved stress tensor.

But this theorem does not validate this equation 1 I just erased. Because this theorem, even though it's very powerful, it does have a hidden assumption. It does have a hidden assumption. This assumption is so obvious and so self-evident, that even though Weinberg and Witten, they're extremely careful people, they did not bother to mention it. In those times, nobody with their right mind would have mentioned such a thing.

So the hidden assumption is that the kind of particles they are talking about-- is that the particles are they talking about, the particles, which they tried to rule out, live in the same spacetime as original theory. OK.

So if you write down, say, QED in Minkowski spacetime, and then you ask about whether you can have some spin-2 particles coming out from QED. And then of course, you ask about whether you have particles come out QED living in the same Minkowski spacetime.

Yeah. So in some sense, this assumption is self-evident. But precisely, this is assumption, which taking advantage by this relation, which first envisioned by [INAUDIBLE], is that precisely in this relation, in this equation 1 which I raised, that the gravity actually does not live in the same spacetime as the original theory. And does not live in the same spacetime as original theory.

So even in the early days, when people were trying to dream having such kind of spin-2 particles that come from QCD like theories, there's already an immediate puzzle.

Suppose the QCD like theories can have emergent graviton in the same spacetime. Then firstly, your QCD theory is defined in a fix spacetime. But now if it contains spin-2 particle in the spacetime itself, it will be dynamical. Then, where does the original QCD theory [INAUDIBLE]? OK.

And then you will just similarly go into this kind of self-contradiction, like somehow-- how can that-- yeah. Anyway. So in this relation, in the holographic duality, gravity does not live in the same spacetime. OK.

So this, in fact, live in one dimension or higher, or they can be several dimension higher depending on the examples. So this void this theorem in obvious way. And also, void this conundrum I said earlier. Because in a different spacetime. So the field theory is still in the fixed spacetime, but then the gravity then can be dynamical. Then, the spacetime which gravity lives can be dynamical.

So any questions regarding this so far? Yes.

AUDIENCE: [INAUDIBLE] in GR, there was no conserved Lorentz-covariant stress tensor. Because the covariant derivative of the stress tensor [INAUDIBLE].

HONG LIU: No covariant derivative is not conserved. This covariant is not conserved. Conserved means the ordinary derivatives, not the covariant derivatives. A covariant conserved stress tensor is not conserved. If you write down a conserved stress tensor, then it's not covariant. Then it's not covariant. Yeah.

Similar thing with this case for the gauge fields. And in this case, gauge invariant currents-- the gauge invariant and the conserved currents is not Lorentz-covariants. And the Lorentz-covariants, gauge invariant current then is not conserved. Any other question?

AUDIENCE: I have a question.

HONG LIU: Yes.

AUDIENCE: [INAUDIBLE]

HONG LIU: Sorry. Say it again?

AUDIENCE: [INAUDIBLE] theory, every particle is massless.

HONG LIU: Yes.

AUDIENCE: And the [INAUDIBLE] spin-2 particle, which carries [INAUDIBLE]

HONG LIU: Yeah. For [INAUDIBLE] field theory, indeed everything's massless. You can construct Lorentz-covariants conserved stress tensor. So that theory [INAUDIBLE] does not allow spin-2 excitations in the same spacetime.

AUDIENCE: [INAUDIBLE]

HONG LIU: There's no massless spin-2 particle.

AUDIENCE: [INAUDIBLE]

HONG LIU: Yeah. Yeah, but you don't have massless spin-2. You don't have massless spin-2 particles.

AUDIENCE: [INAUDIBLE]

HONG LIU: No. It depends. Actually, talking about the massless in the [INAUDIBLE] field theory is a little bit tricky. It's a little bit tricky. They come in all kind of spectrum. They come in the continuous spectrum. Yeah. Yeah, but this theorem does apply to ordinary [INAUDIBLE] field theories. So in those theories, you won't have massless.

Yeah, when we prove the theorem, then you can see precisely what do we mean by massless spin-2 particles. If you stay in the course, then you will know. Yeah. Yeah, we will see it later on. But not for a while. Not for a couple of weeks. And maybe in a couple of weeks, we will see it. In two weeks. One or two weeks. Any other questions? OK. So now, let me give you the proof, which is pretty simple. And use some elementary facts, very elementary facts.

Because you see, this theorem contains very little input. That theorem really contained very, very little input. So if you can prove such a thing, it cannot be complicated. If you can prove it, it cannot be complicated. Because the input is so little. It does not require any details. It does not require any details of a theory. So if you can prove such a theorem, it must be just from kinematics, not come from any dynamics. So indeed, that's how the theorem goes.

So the proof. So we will do proof by contradiction as you would naturally expect. So let us suppose there exists a massless spin-j particle. Massless particles. So let's assume this theory contains some theories [INAUDIBLE] exist massless spin-2 particles. Massless particles of spin-j. And just from the Lorentz symmetry, you can immediately write down-- so from what we learned in QFT 1 and 2. So just by Lorentz symmetry-- it doesn't matter what is the nature of those states. Because the mass formal representations over the, say, Lorentz group and [INAUDIBLE] group.

So [INAUDIBLE] immediately write down the one particle state of such particles. And you normally can write them as k and sigma. So k just is the ordinary spacetime momentum. So for massless particle, of course the k0 squared should be equal to k squared. And the sigma is the helicity. Helicity is a projection of the angular momentum in the direction of your momentum. And for a massless particle of spin-j, the sigma can only be plus minus j. So this is helicity.

So if you don't remember this, you can go back to review Wemberg's book. Wemberg's volume 1 QFT. Volume 1, Section 2.5 discussed how you write down the general one particle states for, say, any quantum-- in any quantum theory as far as you have Lorentz symmetry. OK.

And the one properties of this sigma is that let's now consider rotation. Now, let's consider rotation. So let's consider rotation operator. This [INAUDIBLE] by angle theta around the direction of the k. Around the direction of the k, which is the direction of the k here. So let's consider the action of these under state, this one particle-- this massless particle state.

Then by definition, this just give you i sigma theta k, sigma. So this is essentially come from the definition of the helicity. And the k does not change because this is the rotation along the direction of the k. The k itself does not change. But this eigenstate of this rotation [INAUDIBLE]. OK. Again, if you don't remember this, go back to Wemberg's book and you can easily find it.

So now if you have a conserved current, if you have a covariant conserved current, J mu, then you can always construct the charge, the conserved charge, which is the integral over J0. And similarly, if you have T mu mu, then from T mu mu, you can construct a momentum operator, which is the T mu. So this is just the standard momentum operator in your field theory. So this is just still the setup.

Of course just by definition, the p mu acting on k sigma, you just get back k mu. OK, because this is the momentum eigenstate. And if this particle is also charged under this conserved charge, then-- so if charged under k, under this conserved charge, then Q act on this state will also give you the charge q. So this is operator. This is operator. These are all operator. Operator. And then, this should give you charge q.

So now we want to show this 2 theorem implies-- so with this set up, this theorem can-- so we want to show two things. 1, if q is nonzero, that means if the particle is charged under a conserved charge, then j must be smaller than 1/2. It cannot be greater than 1/2. So we assume it's a conserved stress tensor. Otherwise, j should be always smaller or equal to 1 at most. OK. So that's what that theorem means translated in this language. OK.

Any questions? So now, let's prove it. Let me just do it here. So now, let's prove it.

I think we just have enough time, maybe to prove it. So to prove it, let's just first list some elementary facts. First, just do some elementary fact.

First, from Lorentz symmetry-- we do have to assume Lorentz symmetry. OK. You can show-- so this I will just write down the answer and leave yourself to show it. OK? In your PSET.

So if you look at the matrix [INAUDIBLE] of this conserved current between two such states of different k, in the limit which k goes to k prime, you have q k mu k0.

And similarly, in this limit-- so this we will do it in your PSET. So let me call this equation 2, equation 3. So this is based on the following normalization. Based on the following normalization between the two states. OK.

So this relation is actually very easy to see intuitively. Yeah, you can prove it by-- yeah, let me give me just a quick hint for [INAUDIBLE] regarding 2, is that you can first show that k sigma J0 sandwiched between the two, in the limit when this k become the same is just given by q divided by 2 pi cubed. So this is, again, somewhat intuitive because k is eigenstate of Q. Because the k is an eigenstate of this Q hat. And Q hat is just the integral of J0. It is integral of J0.

And so you can imagine on the right-hand side, there must be a Q because these are essentially the eigenstate of the integration with J0. And then when you take into account all these [INAUDIBLE] functions here, and the integral-- and the relation between the integral, then you will naturally find the right-hand side is just q divided by this.

So once you know this, then you can immediately see the first [INAUDIBLE] equation 2 there. Because at first, there's a j mu there. Then the right-hand side must have index mu. So the only thing can carry index mu is the k mu. There's nothing else.

And also, you know when the mu equal to 0, you should just get the q. Then, downstairs we might have a k0 to cancel them. So this relation, in principle you can just-- just by intuition, you can just-- by dimensional analysis, [INAUDIBLE] just directly write it down.

And similar in the second relation-- in the second relation, the analog of q. So q is the charge associated with Q hat. The analog of the q for the stress tensor is just the k mu. So you just replace this by k mu. It just relates to the q by k mu. Then you get the second line. Because k by k mu [INAUDIBLE]. OK. So I will leave yourself to prove this rigorously.

So this is the second observation. It said for massless particles, which have k squared equal to 0, and k prime squared equal to 0, because [INAUDIBLE] particles. So k squared and k prime squared should be equal to 0.

And that means for any two particles, a massless momentum like this, you see immediately just by writing this down this must be smaller than 0. So this should come from your high school physics. The k dot k prime should be smaller than 0. And that means that k plus k prime must be time-like. OK, must be time-like. Because this square is essentially given by this guy because k square and k prime equal to 0. So this guy must be time-like. OK.

And so that means you can always choose a frame that the special part of this is 0. We can choose a frame k plus k prime equal to 0. So such a frame is just k mu, say E, 0. Say we put all of these in the z-direction, in the 3-direction. Then you can choose k prime k mu equal to-- OK?

And this just follows. These two are two [INAUDIBLE] particles. OK, good. So now, we can almost envision this. Then, there's still a third elementary observation we need to put here.

So the third observation which uses this formula. So let's now take sigma just to be j. So now, under a rotation of theta around 3-direction-- around this direction, which we have the [INAUDIBLE] momentum. So we choose a momentum to be in this direction. Then you should have-- so let me call that operator to be R theta is the rotation in the z-direction, in the 3-direction. So this operate on k, j. So let's take sigma equal to j. Take sigma equal to j. So this should be just equal to exponential i, j theta k, j.

And if this acting on k prime j. So k prime have opposite orientation as k because this is minus e. And the opposite orientation means your helicity should also change sign. So that means here should be minus i j theta k prime, j. OK.

Yeah, because this is the rotation along the positive 3-direction. OK. So now we are in business.

So now finally, we will try to-- from those facts together, we will now see a contradiction. Now see a contradiction.

Now you can see that this object. Consider k, j-- k prime, j. Say this R minus 1 theta acting on J mu R theta k, j. So now, let's consider this object.

So this object, I can evaluate it by two ways. First, I can just act R theta on both states. OK. So R theta on this one just gives me this. R minus 1 acting on that just give me opposite of this. So all together, if I do that way, then I get exponential 2 i j theta k prime, j J mu k, j.

But I can also act this operator on this current. So this current should transform as a Lorentz vector. So if I do this way, then I should get lambda mu new k prime, j J new k, j. So these acting on that should just effect a Lorentzian summation on j. And so this is in the Lorentzian summation matrix. So lambda mu mu just is the standard rotation matrix acting on the vector. So lambda mu mu just the standard rotation matrix. For example, just given by 1, 0, 0, 0. Say, cosine theta minus sine theta. So just a rotation along the 3-direction. So this just the rotation acting on the j.

So now similarly, if you examine this quantity. Yeah, similarly if I just replace here by T mu mu, OK? by T mu mu-- we are out of time, so let me be a little bit faster now.

Similarly, if we replace that by T mu mu, then we will get the same thing 2 i j theta k prime, j T mu mu k, j, which obtained by acting directly on k, j.

By the way, that R acting on T mu mu, then T mu mu transform as a 2 tensor. Then we should get lambda mu rho mu lambda. Lambda mu rho k prime, j T lambda rho k, j.

So now, here's the key. Now, here's the key. So do you already see the contradiction?

So let's look at this equation. This is just like a vector. The whole thing is like a vector. The same vector. So this is like an eigenvalue equation. So this lambda mu mu acting on this vector get eigenvalue 2 i j theta. So this is like an eigenvalue equation.

But if you look at this matrix, this can only have eigenvalue exponential plus minus i theta and the 1. So this is thing we are familiar with. So [INAUDIBLE] minus i theta, 1. So similarly with this equation, just you have 2 lambda, OK?

So now, that means if this quantity is nonzero, which we know it is nonzero from 2 and 3. It is nonzero. That these 2 i j theta can only take a value plus minus i theta and the 1. So that means that j-- from here, let's say equation 4 and 5.

From equation 4, that j must be smaller than 1/2. Because there's a 2j there. Similarly, from equation 5, means j must be smaller than 1. So this is what we want to prove. So this is what we want to prove. Yes?

AUDIENCE: Aren't k and k prime [INAUDIBLE]? I mean, 2 and 3, the limit k goes to k prime and k plus k prime is 0.

HONG LIU: Yeah. So I can take them-- if j is greater than 1/2. For example in this relation, if j is greater than 1/2, then the only possibility is for this to be 0. But we know when you take k equal to k prime, at least in that limit this is nonzero. And so this cannot be 0. So this quantity cannot be identical to 0. OK. So this gives you the proof.

So if you find the loophole in this proof, other than they have live in the same dimension-- yeah, that would be great.

AUDIENCE: What is the whole matrix element is 0? What if the whole matrix element is 0?

HONG LIU: No, the point is that the whole matrix element cannot be 0.

AUDIENCE: Yes.

HONG LIU: Because of this.

AUDIENCE: Isn't the limit k goes to k prime? And there, you assume that k plus k prime is 0. So it means both k and k prime should be 0, no?

HONG LIU: I'm sorry? No. k plus k prime is not--

AUDIENCE: [INAUDIBLE]

HONG LIU: No, this is a spatial vector.

AUDIENCE: [INAUDIBLE]

HONG LIU: No, this is spatial vector. No, this is spatial vector. Yeah. Yeah. This is spatial vector.

OK. Sorry, I'm a little bit late. Yeah. That's all for today.