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**HONG LIU:**

OK, let us start. So last time, we finished discussing quantization of closed strings and open strings. And we see-- we saw that if you want to quantize strings consistently in Minkowski spacetime, then you need 26 dimensions.

We also saw that when you quantize the photon-- when you quantize open strings, then you see a vector field, a massless vector field in the spectrum plus an infinite of [INAUDIBLE]. And when you quantize closed strings, then you see massless spin-2 particles, which is a graviton in your massless spectrum plus infinite number of [INAUDIBLE]. So that essentially is the essence of string theory, is that the strings contain graviton in a quantum mechanical way. So it should be consistent-- should be considered as a serial quantum gravity.

So today, we start talking about another object in string theory. It's called the D-branes. OK? So we discussed open strings, so let's go back to the open string.

For the open string, say after we fixed the [INAUDIBLE] to be flat, the [INAUDIBLE] action can be written as the following. OK? So you can-- when you get the-- when you obtain the equation of motion for  $x$ , you will have something like this. So you need to do an integration by parts plus a total derivative term, or plus a total derivative term until the derivative term we induce a term on the boundary if you have an open string. Say at  $\sigma$  equal to 0 and  $\pi$ . OK?

So this is still  $d\tau$ . And so from this term, you get equation of motion. And then you induce a boundary term, evaluate it. If it's open string, there's two boundaries, at 0 and  $\pi$ . To do such a boundary term at the boundary, we need to also get rid of-- otherwise, the equation of motion is not satisfied.

And what we discussed is [INAUDIBLE] Neumann boundary conditions. It's just taking the  $\pi$  of  $\sigma$  equal to 0,  $\pi$  of  $\sigma \times \mu$  to be 0. But in general, for any  $\mu$ , say for any  $\mu$ , you have a choice of four boundary conditions. You can choose  $\delta x^\mu$  equal to 0, or  $\partial_\sigma x^\mu$  equal to 0. Say and  $\sigma$  equal to 0.

And you have the same choices at  $\sigma$  equal to  $\pi$ . So altogether, you have actually four possible boundary conditions for each direction. So for each direction, you have four possible boundary conditions. If you can see the all possible directions, there are many, many different boundary conditions. OK? There are, in principle, many, many different boundary conditions.

So let's now look at what the other boundary conditions mean. OK? We looked at the case which the-- you just imposed this everywhere, at both  $\sigma$  equal to 0 and  $\sigma$  equal to  $\pi$ , and the four directions. But now let us explore these more general boundary conditions.

First let's just quickly look at what this boundary-- so this is called Dirichlet boundary condition, this is called the Neumann. So first let's look at what does this boundary condition mean. OK?

So if we have-- say let's consider some direction,  $i$ , set  $\delta x^i$  equal to 0. Say at  $\sigma$  equal to 0, say  $\tau$  equal to 0. So this is the example of this Dirichlet boundary condition in the  $i$  direction. Say suppose that  $\sigma$  equal to 0, you have-- you impose this.

So what this means is that the vibration of  $x_i$  at  $\sigma = 0$  at all time does not change. So  $\tau$  is like a parameter I should watch it-- time, the watch it motion. So this tells you that, at all time, the value of the  $x_i$  does not change at the endpoint. So what this means, this directly translates into  $x_i$  at  $\sigma = 0$ , and  $\tau$  should be a constant. OK? Should be a constant.

So this is the Dirichlet condition. And so let's look at some examples to understand a little bit more explicitly what this kind of condition-- what this Dirichlet condition means. OK? So let's first consider the example.

Say suppose in all directions, say from 0 to  $d-2$ , OK, we use-- suppose use Neumann boundary condition as what we did before for both  $\sigma = 0$  and  $\pi$ . But for the last coordinate, [INAUDIBLE]  $x_{d-1}$ , so let's impose the Dirichlet boundary condition.

So let's take it to be  $A$  at  $\sigma = 0$  and  $\pi$ . OK? So for the moment, let's take the boundary condition to be same at both  $\sigma = 0$  and  $\pi$ . So let's look at what did this condition means. Let's look at this example.

So what this condition means-- so let's draw a figure. So, I cannot draw many directions. In particular, I cannot draw  $6 \times 26$  dimensions. So let me call the horizontal axis as all the-- the connection of all the dimensions up to  $d-2$ . And under this axis is the  $d-1$ . OK?

Then the endpoint, the  $x_{d-1} = a$  at the endpoint-- so suppose that this is the line of  $x = a$ . So suppose this is the hypersurface for  $x = a$ , then that boundary condition tells you that whatever open string the motion, the endpoint must be restricted to this hypersurface. OK?

So that's what this-- geometrically, what this condition means. It just says, let's consider  $x_{d-1} = a$ , which is a hypersurface in spacetime, and then this condition means that the endpoint of the string must be restricted to this surface. The interior of the string can be any-- can be out-- can have no restriction, but the endpoint have to be on this surface. OK? Is this clear?

OK. So you can generalize. So this is the first example. You can generalize.

Let's consider  $x$  from 0 to some value,  $p$ . Of course,  $p$  is smaller than the total number of dimensions, which is  $n$  for  $\sigma = 0$  and the  $\pi$ . And then the rest-- say  $x_{p+1}$ ,  $x_{p+2}$ , to  $x_{d-1}$ , these directions we impose the Dirichlet boundary conditions. And then let me call this one back to  $a$ . So I'm back to  $a$ . For both  $\sigma = 0$  and  $\pi$ .

OK? So  $a$  is just some constant vector of the same dimension as the number of  $x$  here. The same as the number of  $x$  here. And this is just a more general way to impose this to more directions. OK?

Is this notation clear? Yes?

**AUDIENCE:** Why is there a repeated superscript?

**HONG LIU:** Oh, sorry. It's  $p+2$ . Yes?

**AUDIENCE:** Maybe do we talk about, like, a different  $\sigma = 0$  behaviors, or is that not interesting?

**HONG LIU:** No, we will talk about that. Let's look at the simple case first. So is this notation clear?

So again, I can do this on the similar plots like this. So now horizontal axis [INAUDIBLE]  $x$  0 to  $p$ , and the vertical axis I do  $p$  plus 1 all the way to  $d$  minus 1. So now those axes are multiple dimensional.

And then this specifies-- so this specifies a  $p$  plus 1 dimensional hypersurface-- surface. So the value of those coordinates are fixed, but the value of those coordinates can be arbitrary. So this  $p$  plus 1 free parameter here. OK?

And so this is a  $p$  plus 1 dimensional surface in the spacetime. And this condition means that the endpoint of whatever open string must be restricted to this surface. OK? Is the picture clear? Yes?

**AUDIENCE:** [INAUDIBLE]

**HONG LIU:** Hmm?

**AUDIENCE:** [INAUDIBLE]

**HONG LIU:** Yeah. We are talking about how to impose the-- what's the meaning of imposing the Dirichlet boundary condition. Instead of Neumann boundary condition. So we now impose the Dirichlet boundary condition, which goes one into the [INAUDIBLE] of the endpoint should be 0. And then this can be specified by the endpoint if you take some constant value. And so that's the geometric way to think about these different Dirichlet boundary conditions.

So if you look at this naively, so previously, when we can see the Neumann boundary condition for all directions, then the string-- then there's no restriction to the string. Then the string can move anywhere. So now when you-- when in some directions we impose the Dirichlet condition, then we see that-- then the motion of the string get restricted, and that this endpoint now have to [INAUDIBLE] some soft dimension of the spacetime. OK?

So at first sight, this may seem to be a nonsensical thing to do. Just say-- you ask why somehow I have to restrict some open strings to some sub-manifold in the spacetime. So why should I do that? In particular, that breaks the Lorentz symmetry. In particular, that breaks Lorentzian.

So the nice thing about the Neumann boundary condition we did before, in more directions, so that respects Lorentz symmetry. And once we introduce some directions to the Dirichlet boundary condition, you necessarily break Lorentz symmetry, because we select out some directions with some others. OK? And so you necessarily break Lorentz symmetry.

You also break translation symmetry because of the-- you have to specify some sub-manifold. You also break the translation symmetry. So at first sight, this seems to be a nonsensical thing to do. So that's why, for many years, actually people did not look at this object, even though for a person to work out string theory on the first day, you should already realize those are the boundary conditions. But for many years, people did not pay attention to this kind of boundary conditions.

It turns out that standard of conditions plays a key role in string theory. And the key is to take a slight different perspective at [INAUDIBLE] sub-manifold. OK?

So now what you should think about is-- and now what you think about is the following. So you think about maybe for the whole spacetime, the whole spacetime, only closed string is the most natural object, because it can move anywhere. But then there's some special places in the spacetime, some kind of defect. And at those defect, the closed string can break to become open string. OK? Can become open string.

And if you take that perspective, then that gives special meaning to such kind of special sub-manifold which open string can end, then there's some kind of object there. Because of the existence of certain object there, then the open string can end there. OK? Yes.

**AUDIENCE:** So indeed, if you have that perspective, then instead of having it lie along hyperplanes, is it more natural to have it just lie upon some other manifold?

**HONG LIU:** Oh, you can-- yeah, I'm just taking you-- I'm just sending you the simplest example.

**AUDIENCE:** Sure, sure.

**HONG LIU:** And once you realize in this simple example, then you can generalize the [INAUDIBLE] complicated scenario you can imagine. Now let's just think about, in the simplest case, imposing those boundary conditions become natural if you think those places in spacetime are special places which something can happen. It's not the same as other places.

And to say those [INAUDIBLE] give a name. So you imagine-- so for example, you can imagine some spacetime defect sitting at, say, this  $x^p$  plus 1,  $x$  equal to  $a$ , sitting at that location. So where open string can end.

So such a defect, we'll call it D-brane. We give a name. So D is clear because it's Dirichlet boundary condition, and the brane because it's some extended surface. OK?

And so we often use a notation to say  $D_p$ -brane. So  $p$  denotes the [INAUDIBLE] spatial dimensions along the brane. OK? So in this case, there's  $p$  spatial dimensions along the plane-- along this brane, so we call it  $D_p$ -brane. OK?

So right now,  $D_p$ -brane is merely just a place which impose a boundary conditions which open strings can end. But we will soon see that actually a  $D_p$ -brane is a dynamic object, and it's actually a dynamic object. And this actually can move, can fluctuate, can jump around, et cetera. OK?

And so also from this perspective, our previous Neumann boundary condition, if we have Neumann in all direction, so if we have Neumann boundary condition in all directions, then in this case the open string can end everywhere. End anywhere.

From this new perspective, we call this space-filling brane. We call this actually a space-filling brane. OK? So the Dirichlet boundary-- the Neumann boundary condition we considered before should be considered as actually you fill the whole spacetime with such a brane, and then the string can, in principle, move anywhere.

So for example, in the bosonic string we talked about before, this group because one and two are D 25 brane, so D 25 brane because there's 25 spatial dimensions. OK? Such a brane when you impose the Neumann boundary condition in all directions. Or if you are in the super string, which the total number of spacetime is 10, and then the spatial number-- spatial dimension is nine, then you have D9-- then these other D9 branes.

Any questions about this? Yes?

**AUDIENCE:** [INAUDIBLE] boundary conditions, they're imposed at the quantum level, right? It's not just that the classical solution has to have its endpoints on this D-brane, but it's the fluctuations around it [INAUDIBLE] solution that we consider?

**HONG LIU:** We will talk about that. We will talk about that. Any other questions? Good.

So let me make a note. You can, in principle, impose the Dirichlet boundary condition in any direction you like, except in the  $x_0$  direction. OK?

It's because  $x_0$  cannot be a constant.  $x_0$  have-- so  $x_0$  is a time, is our real time in the Minkowski space. So the string moves in this Minkowski spacetime, and this  $x_0$  is that time. And the time cannot stand still. OK?

And in order to have this, you needed that location to stand still. And the time cannot stand still. So this cannot be a constant. So you cannot impose the Dirichlet boundary condition in the time direction.

So the smallest in the Minkowski spacetime, the smallest would be at D0 brane, which we impose the Dirichlet boundary condition in all spatial directions except in time. In time, you always impose the Neumann. Then the smallest situation will be called D0 brane, which is a point. And then the open string can end on this point. OK? Yes?

**AUDIENCE:** So I have question. So this idea about calling the Neumann boundary conditions to D25 branes or D9 branes. So this is a little bit perplexing, because it seems like not imposing Dirichlet boundary conditions, this implies that not imposing Dirichlet boundary conditions is equivalent to applying Neumann boundary conditions, which is not-- they're not, like-- I mean, they're like mutually exclusive. It's not like-- so I mean, I don't understand why it is that-- so does this thing behave like a 24 brane? Like, does it have the same properties as, like, a 24 brane or something like that?

**HONG LIU:** Sorry, which? You mean 25 brane.

**AUDIENCE:** So what I'm saying is that it seems that the Neumann boundary conditions have a different character than-- like a different flavor of how they work than the Dirichlet boundary conditions. So it just seems strange to me that you would still call it a brane, like a D 25 brane when you impose them or not.

**HONG LIU:** No. So in those cases, in those cases-- for example, in this case, which we Neumann boundary conditions in all the other directions except one, then this is a 24 dimensional-- then this is a 25 dimensional manifold which the string can end on. And I just--

**AUDIENCE:** Oh, I see, of course. So, right. You have to apply the Neumann boundary conditions to all of the other [INAUDIBLE]. OK, I understand. Yeah.

**HONG LIU:** Yeah, yeah, yeah. Yeah, so it's-- yeah. Any other questions? OK, good.

So now let's consider more complicated-- slightly more complicated boundary conditions. So by introducing this D-brane, now you can actually have all different possible boundary conditions. As we said here, there-- for each direction, you can have in principle four possible choices. And then if you have [INAUDIBLE], so for time direction you don't. But for the 25 dimensions, 25 spatial dimensions. And each dimension, you have four choices, and you have four to the power 25 possible choices of different boundary conditions.

Now let's talk about the other boundary conditions. So by using the picture of D-brane, we can talk about them all. We can talk about them all.

So then let's consider model 1 branes. OK? So let's consider the situation we have two such Dp-brane. So this direction is still  $x_1 \dots x_{p-1}$ . And then this is all the other directions.

So suppose we have a brane at  $a$  and another brane at  $b$ , at the vector specified by  $b$ . OK? So now in this case, we can have four types of open strings.

We can have four types of open strings. For example, I can have an open string which ends at  $b$ . I'm going to have an open string end on  $a$ . But then I can also have an open string which goes from  $b$  to  $a$ , and for open string from  $a$  to  $b$ . OK?

So in this case, the boundary condition would be the  $x_{p-1}$  to  $p$  is still a Neumann boundary condition for both  $\sigma = 0$  and  $\sigma = \pi$ . But for the other directions, you can have either-- your  $\sigma = 0$  can either be in  $a$  or  $b$ . Or  $\sigma = \pi$ , you could do  $\pi$  can either be  $a$  or  $b$ .

So that gives you the four types of strings. OK? That gives you four types of different strings.

Now you also can see the branes of different dimensions. So now you can see the branes of [INAUDIBLE]. So for example, say suppose let's think of the two directions,  $D-2$  and  $D-1$ . And let's suppose all the other directions are in the Neumann direction.

So this can see the two kinds of branes. One is extended in the  $D-2$  direction [INAUDIBLE] some value of  $D-1$ . And then you can see another brane, which is localized in both  $D-1$  and  $D-2$ . OK?

Then you can again have four types of strings. [INAUDIBLE] and the ending between these two. So again, you can see the [INAUDIBLE] string because we can see the [INAUDIBLE] string. So you can again see the string like that.

So let me call this brane 1, this brane 2. So let's consider the 1-2 string. It reaches from 1 to 2. Then the 1-2 string, with the corresponding to the boundary condition, say at  $D-2$ , say at  $\sigma = 0$ , you impose the Neumann boundary condition because it can move freely in the  $D-2$  direction.

And but in the-- at  $\sigma = \pi$ , then this should be fixed at some value. So let's suppose, say, this is  $a_1$ ,  $b_1$ . Then add to that, we are fixed at  $a_1$ . OK? So this is 1-2 string.

Then you can have one end which is a Neumann boundary condition, but the other end which is Dirichlet boundary condition. OK? Is this clear to you? And then for this 2-1 string that will be opposite, for the 2-1 string for the  $D-2$  direction at  $\sigma = 0$  will be the Dirichlet condition, but at  $\sigma = \pi$  would be the Neumann condition.

Any questions about this? So these are pretty simple. You just need to get used to it and spend a couple minutes drawing a couple of diagrams. Then you will be an expert. You will be an expert. This is pretty elementary.

So important remark. So a Dp-brane, as we said before, it breaks translation symmetry. Translation and the Lorentz symmetries. OK? Of the original Minkowski spacetime to your sub-group.

So the subgroup is that you have-- still have translational symmetry and the Lorentz symmetry along the brane direction. So along the brane direction, you still can translate-- your string can still translate in that direction. And there's still Lorentz symmetry in that direction.

So there's still Poincare symmetry in  $1, p-1$  comma  $p$ -- so in the time and the  $p$  spatial dimensions along the brane. And then you have rotational symmetries in the direction transverse to the brane. So that's  $D-1$  minus  $p$ .

So that's the rotation among those coordinates. So you can have the Lorentz symmetry along those directions, then the rotational symmetry is along those directions. OK? So this is for  $D_p$ -brane.

So this is the remaining symmetry. So this is remaining symmetry say if you have a  $D_p$ -brane in the spacetime. Of course, if you have multiple  $D_p$ -brane, et cetera, then it's broken even more. Yes?

**AUDIENCE:** [INAUDIBLE] branes are [INAUDIBLE] open not [INAUDIBLE].

**HONG LIU:** Oh. Yeah, we can talk-- here, I'm just talking about the simplest situation. I'm imposing the simplest boundary condition, just like these kind of boundary conditions. And then these kind of boundary conditions is-- applies to the infinite-- yeah, you can-- once later we talk about the  $D$ -brane can be dynamically excited, then you can see the complicated configurations. Yes?

**AUDIENCE:** So [INAUDIBLE] boundary condition [INAUDIBLE].

**HONG LIU:** Yeah.

**AUDIENCE:** [INAUDIBLE]  $x$  plus [INAUDIBLE].

**HONG LIU:** Ah. You are saying something like this-- you're saying some combination of this?

**AUDIENCE:** [INAUDIBLE]  $x$  plus [INAUDIBLE].

**HONG LIU:** Yeah, yeah, yeah. That can happen. That does not happen in the straight Minkowski space, because in the straight Minkowski space this is the action. So you only have one or the other. But if you, say, have Minkowski but you turn on some electric field, then you can actually have such kind of mixed boundary conditions.

Yeah. Just they exist-- it depends on the specific dynamical situation. So this is empty, flat Minkowski space, then you can only have those boundary conditions.

**AUDIENCE:** And when [INAUDIBLE]?

**HONG LIU:** Sorry?

**AUDIENCE:** When [INAUDIBLE].

**HONG LIU:** That I'm not sure. That is different. Yeah, because that does not satisfy that boundary condition. And so we have a look at the specific situation. We have to look at the specific situation. Just for the flat Minkowski space, these are the only boundary conditions you can have.

And we will not have-- we will not be able to do it in this course, but you can see the situation because of the-- in open string, you can have electromagnetic field. You can have a photon. So you can, in principle, turn on electromagnetic fields. And then we modify the boundary condition. Then you can have actually mixed boundary conditions.

**AUDIENCE:** Do people ever consider this is a [INAUDIBLE] between two D-branes as a [INAUDIBLE] defect? Because [INAUDIBLE] between--

**HONG LIU:** You mean string between different-- string before-- between different D-branes.

**AUDIENCE:** Yes.

**HONG LIU:** You can see that the D-branes themselves have topological defect, rather than the string between them has the topological defect. The D-branes themselves are defects. Strings are just-- as we will talk about-- we will talk about it now, the strings on the D-brane, they're just the excitations of the D-branes.

So now after talking about those boundary conditions, then we can again go through the procedure of quantizing open strings with those boundary conditions. Then we see what do we get. Now let's talk about the open string spectrum on the  $D_p$ -brane.

So as you can see already from those pictures, because of the restriction on the endpoints, so essentially those strings, they can only move along the direction of the brane. They can only move in the direction along the brane.

So essentially, they can be considered as living on the brane. OK? So those open strings are living on those  $D_p$ -branes. OK? So when we quantize those open strings, then we will get the excitations of those open strings, the state of those open strings, on the D-branes. OK?

So let me first introduce some notations. Previously, we have the notation  $\mu$  refers to all directions. So now I separate into [INAUDIBLE] and  $a$ . And [INAUDIBLE] is the Neumann boundary-- Neumann directions from 0 to  $p$ . And the  $a$  are the Dirichlet directions from  $p+1$  to  $D-1$ .

So there's  $D-1-p$  of them. So  $D-1-p$  of them. So for the Neumann boundary condition, it's just identical to what we had before. So let me just write it down to remind you.

OK. So these are the, say, the most general solution. So in this case in particular, if you remember you have  $x_a$  you could do  $x_r$ . So the left moving and right moving are the same, controlled by the same function. And so that's why you get this cosine  $\sigma$  cosine. And when you take the derivative of a  $\sigma$  becomes sine, then vanishes at the endpoints. OK?

So for this story, it's just exactly identical as before. So now let's look at the Dirichlet conditions. So for the Dirichlet conditions, let's again write down the most general possible boundary conditions for Dirichlet boundary-- yeah, first write down the most general solution for such a field. So you have  $x_a$  then  $2i p a \tau$ . Then you have [INAUDIBLE]  $\tau$  [INAUDIBLE]  $\sigma$  and [INAUDIBLE]  $\sigma$ . OK?

But now the boundary condition we need to impose is  $x_a$  at  $\sigma = 0$  and  $\pi$ .  $\sigma = 0$  and  $\sigma = \pi$  equal to  $b_a$ . So here, I call them  $a$ , but let me call them  $b$  because [INAUDIBLE] write  $a$  to the index  $a$ . So let me just them  $b$ , the vector  $b$ . OK?

So now we have to impose this boundary condition. So you can immediately see that because of this boundary condition  $x_a$  have to be  $b_a$ . OK,  $x_a$  have to be  $b_a$ . And this  $p_a$  have to be 0.  $p_a$  have to be 0, otherwise you won't satisfy this condition, because this changes with time.

And then you want the sum of them to be 0 at  $x$  equal to  $-\sigma$  equal to 0. So that gives you  $x_0$  equal to [INAUDIBLE]  $x$ . So this is from the boundary condition at  $\sigma$  equal to 0. OK?

And then the boundary condition is  $\sigma$  equal to  $\pi$ , you deduce both of them are periodic boundary conditions. They are periodic functions in  $2\pi$ . OK? They're periodic in  $2\pi$ .

You can see it easily. So at the  $\tau$ --  $\sigma$  equal to 0, we just have  $x$   $\tau$  plus  $x_0$   $\tau$  equal to-- yeah, it should be equal to 0, so you  $x_0$  equal to [INAUDIBLE]  $x$ . So now, you can just write down explicitly just-- still, this is a periodic function in  $2\pi$ , so the expansion is essentially the same.

So the only difference for those directions is that now you have  $b$   $\alpha$ ,  $b\alpha$ . Now you don't have this term, because this term is 0. And then  $a$  not equal to 0  $\alpha$  [INAUDIBLE] divided by  $n$  minus [INAUDIBLE]  $\tau$  sine in  $\sigma$ .

OK, there's some pre-factors here I forgot to write. Some  $\alpha$  prime [INAUDIBLE] by convention, let's not worry about those things. OK?

So now you do quantization. So the quantization in Neumann boundary condition, of course identical to before-- oh yeah, first let me just explain the  $p_a$  equal 0. I just want to make one remark.

So this is, of course, expected. Because if the string is limited in moving on this plane, then of course it cannot move in the perpendicular direction, so cannot have central mass, momentum in the-- so cannot have central mass, momentum in the perpendicular direction.

So this is indeed what we intuitively expect. OK? The central mass momentum can only be in the one direction, or can only be in the Neumann directions.

So the quantization of the Neumann boundary condition is exactly the same as before. And actually, quantization in Dirichlet directions which this is set up also the same, also exactly the same. Because essentially the only change is cosine  $\sigma$  goes to sine  $\sigma$ . You can walk through that procedure, and you find that nothing really changes. So it's exactly the same as that of  $n$ . OK?

In particular, the commutation relations between all those modes does not change at all, just [INAUDIBLE] exactly the same as what you would have in the Neumann case. So this is also [INAUDIBLE]. This is [INAUDIBLE]. So imposing such boundary conditions, essentially you're only restricted to the central mass motion, so you can no longer move in the perpendicular directions.

But the string can oscillate. But string can still oscillate in all 10 dimensions. There's no difference in the way the string oscillates. And so strings still oscillate in 10 dimensions, so everything is the same. Yeah. A string is still oscillating in all directions.

In particular, the zero point energy is the same. Zero point energy on the worksheet is the same. OK?

So this means that the mass formula we derived before for the Neumann boundary condition will exactly apply without any change. OK? Without any change.

In particular, the mass of the spectrum will also be essentially-- the mass of the spectrum will also be essentially exactly the same as before. One second.

So let me just mention one point, that this is identical. This is identical. This is actually related to that for the Dirichlet boundary condition on both sides. This is still a periodic boundary-- a periodic function in  $2\pi$ . And so you get this.

In such a case, say if one endpoint is the Neumann boundary condition but the other endpoint is a Dirichlet boundary condition, then the story is a little bit more subtle. OK? And then you actually get to the-- then the frequency of each modes become half integer rather than integer. And then you will get the difference [INAUDIBLE], and there the spectrum will change, et cetera.

OK? So I will not do it now. Maybe I will put it in your [INAUDIBLE] p set, so that you can do it yourself. Yeah? You have a question?

**AUDIENCE:** [INAUDIBLE] question was what happens when p equals 0 [INAUDIBLE].

**HONG LIU:** Oh, yeah. Yeah, for the p equal to 0, yeah, indeed it's a little bit tricky to do the [INAUDIBLE]. Yeah. Yeah, for p equal to 0, it's tricky to do the [INAUDIBLE]. But you can do-- using other quantization procedures, then you can do the same thing. Yeah, but the conclusion does not change between different p's.

**AUDIENCE:** [INAUDIBLE].

**HONG LIU:** Yes?

**AUDIENCE:** Maybe I missed this, but for the two different brane that you have a string going from one brane to another--

**HONG LIU:** Yeah.

**AUDIENCE:** You said from a point to a brane, the story changes. What about, like, from one brane to another? Is it exactly the same as this, or is it--

**HONG LIU:** Oh yeah, that's also a good question, which I might already have something like this in the current p set. If not, it's something I'll put in the next p set because you asked.

Yeah, you see there's a distance between them. So now you actually need to include the sigma term. Now you actually need to include the sigma term here to account this difference. Because now the 0 and the pi is no longer the same. Yeah, here I don't need to put in the sigma term because the 0 and the pi are the same.

**AUDIENCE:** So the zero point would change--

**HONG LIU:** Yeah, zero point energy will change. And essentially, it's given by the mass of the strings stretched between them. Yeah, so that's-- yeah, that's a very instructive exercise. Actually maybe I already put in the current p set. I don't remember.

Anyway. Other questions? Good.

So now let's look at the-- more explicitly at the open string spectrum. So again, it's the same thing. You just [INAUDIBLE] and  $1, \mu_1$ . You can act  $m_1$  times  $n, k, \mu_k, m_k$ .

The only difference is now your vacuum state is only-- can only have momentum in the p alpha direction. That's essentially the only difference. OK? And again, the mass square now it just matters p alpha-- p alpha. OK, only around the Neumann directions.

So this is really-- then each mode on the string-- each oscillation mode of the string now should really be corresponding to a particle or field in  $p + 1$  dimensional world-volume of the  $D_p$ -brane. OK?

So we call-- this is the world-volume. We call the directions, all these directions together, as the world-volume of the  $D_p$ -brane. So since the string can only move along the  $D_p$ -brane, so they can only correspond into the field of particles within the  $D_p$ -brane. They cannot move in the opposite direction. They cannot move in the perpendicular direction.

OK? So [INAUDIBLE] lives on the brane. OK? And in this-- then all of those things may be considered as the excitations of the branes. OK?

**AUDIENCE:** [INAUDIBLE]

**HONG LIU:** Hm?

**AUDIENCE:** [INAUDIBLE] string can be part of the excitation.

**HONG LIU:** Yeah, that's right. Yeah. Because you can excite certain strings and not excite strings, et cetera. And they're all moving along the brane.

And the important thing is that the states should fall into repetitions now of this symmetry, because that's the only remaining symmetry. [INAUDIBLE] when  $p$  under the rotation in the transverse directions. OK?

And again, just by this, you can show that it actually makes sense only in the 26th dimension. OK? Again, makes sense only in the 26th dimension.

So now let's again look at the massless states. So as I mentioned before, the mass [INAUDIBLE] exactly the same as before, because the oscillation modes behaves exactly the same. So the mass [INAUDIBLE] exact behaves the same as before. So the massless states, that is the following. Yeah?

**AUDIENCE:** When we say the particles in [INAUDIBLE] dimensional world-volume--

**HONG LIU:** Yeah.

**AUDIENCE:** But [INAUDIBLE] particles [INAUDIBLE] object, whereas the open string can-- you know, one end of the open string located at one point [INAUDIBLE] the other one [INAUDIBLE] very far away. But is that picture [INAUDIBLE]?

**HONG LIU:** No. That's a good question. So when I say particles, I'm talking about the center of mass motion. OK? Therefore, each oscillation modes that you see, that's corresponding to-- yeah, for particular oscillation modes, then the-- other than those quantum number-- yeah, let me finish this. I will answer that question. I will write this down explicitly.

So let's talk about massless states. So again, using [INAUDIBLE]. So again, we are using the [INAUDIBLE]. You can separate the [INAUDIBLE] and  $i$  in the Neumann direction. And in the Dirichlet direction, again it's just the  $p + 1$  to  $D - 1$ . OK?

And then the massless modes would be the same as before. You have [INAUDIBLE]  $1i$  acting on  $p$  alpha. And the  $\alpha - 1$  acting on  $p$  alpha. And the plus infinite number. Plus infinite number of massless modes. OK?

So let me just illustrate this point. So this [INAUDIBLE] as before, the [INAUDIBLE] further breaks this Poincare symmetry to only a sub-group. But you can already see from here this must be corresponding to a vector particle. So this is like a massless vector. Massless vector.

Yeah, so to answer your question, so example, for typical modes like this, the only quantum number you have is  $p_\alpha$  and this vector indices. This vector index. So from the perspective of the quantum number-- and this really is just like a vector particle. Just like a vector particle. And then you can describe it by a vector field, et cetera.

It is very good question. Say suppose in this situation which I have a string, suppose a string is really very known. It's very big. Then I should be able to see this one-dimensional structure.

Indeed, then that means that, even though this corresponding-- this is just corresponding to one particular mode of the string. And to see the string structure you have to be able to see, you can excite this into other different modes, et cetera.

**AUDIENCE:** [INAUDIBLE] stretch the string.

**HONG LIU:** Yeah.

**AUDIENCE:** I mean, [INAUDIBLE]--

**HONG LIU:** Yeah, stretching the string corresponding to exciting the string in a different way. It means leaving this particular state.

**AUDIENCE:** Oh, I see.

**HONG LIU:** Yeah. I'm talking about the interpretation of this particular state, which is [INAUDIBLE] spacetime particle.

**AUDIENCE:** Actually this data conversely decided the other configuration of--

**HONG LIU:** Yeah. It determines all the configuration. It determines all the oscillation modes of the open string. Just the other oscillation modes, they are all not excited.

But of course, if when [INAUDIBLE] this one-dimensional object [INAUDIBLE] find the ways to excite them. And then you know this is not a particle, it's a one-dimensional object, et cetera. But in order to see this one-dimensional structure, you have to probe the final structure that just loads quantum numbers. If those are the only quantum numbers you can probe, then [INAUDIBLE] particle.

So this is the massless vector. But now those guys, they have index  $a$ , which is not in the Poincare here. So those are now scalar fields. OK? Those are scalar fields on the world-volume of the D-branes. OK?

So now we find-- so previously, when we can see the all Neumann directions, we only see a vector field. So moving more 26th dimension. But now-- so these massless modes, they give rise to a vector field, the massless vector field  $A_\alpha$ , and  $D - p - 1$  scalar fields  $\phi_a$ . OK?

So essentially, when you introduce D-brane compared to the space-fitting brane cage, which we can see the before in the Neumann direction, just some component of those cage fields now become the oscillation in the transverse direction, and now they just essentially become scalar fields. OK? Is this clear? So now instead you have a smaller vector field, plus a bunch of scalar fields.

As we said before, [INAUDIBLE] defective action, so the [INAUDIBLE] defective action for such massless modes must be-- have the structure, say-- so this is in the  $p + 1$  dimensional world-volume of the string, say, of a Maxwell series. And they're just the same as with scalar fields. And then of course, they are more complicated contributions. Just this is a no energy. Then the structure of those excitations, they must be described by this kind of low-energy factor series. And this is, again, just determined by the universality of massless modes.

So now there's a very-- so now let me-- the first node-- let me emphasize this point, is that the number of massless modes, massless scalar fields. So we have such a bunch of scalar fields. It's exactly the number of transverse directions to the D-brane. OK?

So essentially, associated with each transverse direction you have scalar excitations. You have massless scalar fields. OK? So let me emphasize this point.

So those scalar fields, some of you maybe already have guessed, they have a very simple physical explanation-- they have a very simple physical interpretation. And those scalar fields, they can be considered as describing the motions and the fluctuations of the D-brane in the transverse directions. OK?

So for each transverse direction, you have such massless scalar fields. And each of scalar fields-- so more precisely, so more explicitly, just as a cartoon picture, suppose let's consider, say, the  $p - 1$  direction,  $p + 1$  direction. This direction,  $p + 1$  direction.

So suppose you have a D-brane sitting here at some value of  $a$ . OK? So when we impose the Dirichlet boundary condition-- yeah, so those are the  $x_1$  to  $x_p$  directions. OK? Yeah. Yeah, this is the other  $x_0$  and  $p$  directions. You have a D-brane sitting here at some value of  $a$ .

So when we impose the Dirichlet boundary conditions, that boundary condition does not depend on the value of  $x_0$  and  $p$ . So they're the same everywhere. So that's why I draw it flat, exact flat.

But now, if you have some excitations in the  $\phi$   $p + 1$  direction, so  $\phi$   $p + 1$  is a field in the world-volume of the half-- can have  $p$  alpha momentum. You can go to coordinate space. Then their function's just  $x_1$ ,  $x_p$ . OK?  $x_i$ ,  $x_p$ .

And then the profile of this  $\phi$  essentially tells you that the D-brane-- say if you have a scalar field excitation like this, then the shape of the D-brane actually is no longer strictly flat. Actually have a profile given by these scalar fields. So we are not justify this precise-- so this can be justified mathematically to really show that this is the case. OK? And I will not do that here, but let me just mention some support for this picture. So-- OK.

So first you can show at [INAUDIBLE] level having such excitations-- OK? So having such excitations, so we can just call it [INAUDIBLE] excitations. So this is a picture in momentum space. [INAUDIBLE] excitations modifies the Dirichlet boundary condition. OK? So this you can show explicitly.

So this is first point, which I will not go into there. And the second point is that you can actually work out the precise pre-factor. There's a way, say by doing string theory calculation, say by computing the [INAUDIBLE], there's a way you can actually work out the pre-factor, precise pre-factor.

So let me call this the equation star. So you can actually work out-- so that was just by general-- this is just by general principle. Say if you have a massless spectrum, then the [INAUDIBLE] must behave like this. But this pre-factor is not fixed. OK?

But you can actually, with the interpretation phi, [INAUDIBLE] fluctuations of this Dirichlet boundary condition, you can actually fix the precise pre-factor here. So working out the [INAUDIBLE] you can show that actually this action can be written as the following.  $T_p$ -- again, it's integrated only over the [INAUDIBLE] of the D-branes,  $p$  plus 1.

And actually first we find actually there's a constant term. We will call that action precisely. And then you have this  $F$  term under this-- and let me-- yeah, actually I should have done it there. So there is a summation  $a$  over here, right?

So this  $T_p$ , this pre-factor, is actually-- we can work out the precise pre-factor. And this pre-factor is given by the mass of our  $D_p$ -brane per unit volume. Or in other words, the brane tension. So I will explain where does this come from, where-- how do we define brane tension. But for now, let me just introduce this concept of brane tension.

So if you work out the pre-factor, you find the pre-factor is precisely the brane tension, OK? Brane tension. So now if you look at this equation, so then this 1, it's just like a static mass term. This 1 is like a static mass term. So you also find this 1 in this [INAUDIBLE] action. But this 1 is very important because of the-- so this 1 is like static term.

And now if you look at this term, so now if you consider the situation which  $A_\alpha$  is equal to 0-- so you don't have an excitation of the vector field, but  $\phi_a$  only depend on  $t$ . Only depend on the time. And then this effective action then will become  $T_p$ , then the volume of the  $p$ , then  $dt \frac{1}{2} \dot{\phi}_a^2$ . OK? And so  $V_p$  is the volume of  $D_p$ -brane. Volume of the  $D_p$ -brane.

And this tension times  $D_p$ -brane-- so this is just the mass. So this is just the mass. So you can rewrite this just  $\frac{1}{2}$  and  $p$ -brane  $\dot{\phi}_a^2$ . And then  $D$ -brane [INAUDIBLE]  $T_p$  times the volume.

So you see, this is precisely like a particle. Like the whole brane moving-- the [INAUDIBLE] motion moving of some object. OK? So again, this give you the interpretation that the phi can be interpreted as the [INAUDIBLE] describe the motion of the brane. OK?

OK. So do you have any questions about this? Yes?

**AUDIENCE:** [INAUDIBLE] introduce the dynamics of brane from [INAUDIBLE].

**HONG LIU:** Sorry?

**AUDIENCE:** If we [INAUDIBLE] brane [INAUDIBLE], then with the pre-factors [INAUDIBLE].

**HONG LIU:** No. Those pre-factors depend on your normalization. Those pre-factors depend on normalization. So you can normalize them in some particular way. You can normalize them-- here, what I'm saying is that there is a specific normalization, which then the  $\phi$  can precisely be interpreted as the motion of the D-brane-- yeah,  $\phi$  always corresponding to the motion of the D-brane, but this specific normalization in  $\phi$  can be really interpreted as a dynamical version of this coordinate in the transverse direction. And with that normalization, then you see that this [INAUDIBLE] action is like the motion of a massive object.

OK? It's like the motion of a massive object. And this gives you the confirmation that the  $\phi$  indeed is degrees freedom-- should be considered degrees freedom to describe the motion of the D-brane. Yes?

**AUDIENCE:** That's-- I mean, that is only in the [INAUDIBLE]. If that analogy continued, [INAUDIBLE] higher excitations of those closed-- open strings?

**HONG LIU:** Yeah. So that's a good question, which I don't have time to discuss now but I can-- but I will describe at the beginning of next lecture. So this is a no-energy story. Then you can actually include a subset of higher-order terms. Then you find that there's an infinite number of higher-order terms you can sum, and then this precisely generalize to the motion of a [INAUDIBLE] object.

Yeah, because if you want to-- yeah, because this is no-energy [INAUDIBLE] have to be non-relativistic. But then you can generalize, generally if you sum over an infinite number of higher terms, then this describes [INAUDIBLE] motion of object. And again, that's another independent confirmation that this is the-- describe the motion of the D-brane. Yes?

**AUDIENCE:** So I'm kind of fundamentally confused about how-- so we started with a D-brane which was a topological defect in our background manifold.

**HONG LIU:** Yeah.

**AUDIENCE:** So like a [INAUDIBLE] or something.

**HONG LIU:** Yeah.

**AUDIENCE:** And we are looking at motion of strings in this background manifold.

**HONG LIU:** Yeah.

**AUDIENCE:** And this is-- so why do we all of a sudden have the string motion affect the background? Like we never even introduced any coupling to the background.

**HONG LIU:** Yeah. So that's precisely-- so this is a very good question. This is precisely the remarkable thing about this string theory. So yeah, so I was going to say at the beginning of next lecture, but now let me say it.

So at the beginning, you introduce the D-brane just by some rigid boundary conditions. Some rigid boundary conditions. And looks like this is a non-dynamical object because you are introducing some rigid boundary conditions.

But now, when you quantize the strings under D-branes, and then you find that those strings now give the D-brane itself [INAUDIBLE].

**AUDIENCE:** Is it because there's a graviton here?

**HONG LIU:** No, it has nothing to do with gravitons. [INAUDIBLE] do with graviton. Generally, you start with rigid boundary conditions. But now you find [INAUDIBLE] boundary conditions. Then you can generate-- then you have some excitations. Then the coherent motions of these excitations now give this object dynamics. So that's a precisely remarkable thing about string theory.

**AUDIENCE:** What if you have two strings both ending on the same brane?

**HONG LIU:** No, no, no. No, this is including all the strings. You can have as many strings as you want.

**AUDIENCE:** Well I mean, we're quantizing one particular object, and we're looking at a spectrum of a single string starting here--

**HONG LIU:** Yeah, each string have such excitations.

**AUDIENCE:** But then each different string will generate a different dynamics for the--

**HONG LIU:** No, they're the same dynamic. We just have two particles. It's just like we have one photon here, another photon there. It's the same thing.

And this is the same thing about how string theory generates gravity. So let me just say a few words here. So at the beginning, when we quantize the closed string, we can see the closed string moving, again, in the rigid Minkowski spacetime. Yeah, because that [INAUDIBLE] region. We just consider in the region in Minkowski spacetime.

But now, when you quantize that closed string, when you see that the closed string actually half spin-2 two excitations. Half spin-2 excitations. That means that string actually has gravitational effect. Unless you have spin-2 excitations, then Einstein-- already from Einstein, gravity tells us that the spacetime is no longer rigid, that actually you can deform the Minkowski spacetime by putting in those excitations, these spin-2 excitations.

And if you put enough of them, then [INAUDIBLE] deform the Minkowski spacetime to some other curve spacetime, then you can actually describe generically the quantum gravitational dynamics of a general curved spacetime. So here, it's the same thing. We start with a reach of the D-brane. Then we quantize the excitations-- then we quantize the degrees freedom on the D-brane. And then we find that those degrees freedom corresponding to the fluctuation of the D-brane.

And if you have enough of them-- say if you have  $\phi$  only depend on  $p$  to be the same everywhere, so that's some coherent motion for every point on this brane. So then if you build enough [INAUDIBLE] excitations, then you can actually make the D-brane move [INAUDIBLE] become a fully dynamic object. So that's precisely the remarkable thing about string theory. Yes?

**AUDIENCE:** [INAUDIBLE]

**HONG LIU:**

No, we don't know how to describe a dynamical brane from the beginning. No, that's how we do it. We first start with some-- so this is called a background dependent quantization. You start with a specific solution. You start with a specific configuration of the D-brane, which you happen to be able to quantize. Then you can find all the excitations around that particular configuration, and then you can turn on those excitations to move away from those points.

So that's essentially what we're doing here. OK. Let's stop.