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**PROFESSOR:** OK, let us start. So maybe start by reminding you what we did in the last lecture. So you take the black hole metric so we explain that you can actually extend-- so this is the other black hole horizon-- so you can actually extend the region outside the horizon to actually four total regions for the four black hole spacetime.

And then this [? tag, ?] this is a singularity. And of course, this is only the  $rt$  plane, and then you also have [?  $s^2$ . ?] So now [INAUDIBLE], let me show that in order for the metric to be regular at the horizon into this when you go to Euclidean signature then you want to identify  $\tau$  to be periodic with this period.

So in essence, when you go to Euclidean signature, then this  $rt$  plane becomes, essentially, a disc. And this  $\tau$  is in the angular direction, and then the  $\rho$ , this  $r$  is in the radial direction. And then the region of this disc is the horizon. The region of the disc is the horizon.

So when you say, yeah, so this  $\tau$  has to be periodic. So that means if you fit any theories in this spacetime, then that theory must be at a finite temperature with inverse of this, which gives you this [?  $H$  ?]  $\kappa$  divided by  $\beta$ . And the  $\kappa$  is the surface gravity.

So you can do a similar thing with the [INAUDIBLE]. And the [INAUDIBLE] is just  $1/4$  of a Minkowski, which [? forniated ?] in this way. This is a constant of the  $\rho$  surface, and this is a constant of the  $\eta$  surface.

And, again, [INAUDIBLE] in  $\eta$  to Euclidean. And in order for the Euclidean need [? congestion ?] to be regular, the region, then the  $\theta$  have to be periodical in  $2\pi$ . And then this  $\theta$  essentially, again, becomes an angular direction. And then this implies that the local observer in the window of spacetime must observe a temperature given by this formula, OK? Any questions regarding that?

Good, so today we will talk about the physical interpretation of this temperature. OK? So let me just write down what this temperature means in words, so I will use black hole as an example. We can stay exactly parallel [INAUDIBLE].

It says if you consider a quantum fields theory in the black hole spacetime, then the vacuum state-- so when you put the quantum fields theory, say, in the spacetime, always ask, what is the vacuum? Then the vacuum state for this curve  $t$  obtained via the analytic continuation procedure where the analytic continuation procedure from Euclidean signature is a thermal equilibrium state with the stated temperature, OK?

So emphasize when you want to talk about temperature, you first have to specify what is the time you use. So this temperature refers to this particular choice of time, and so this is the temperature corresponding to an observer while using this time. And then this is the time, as we said, corresponding to observer leaving at infinity. And so this would be the temperature observed by the infinity.

So similarly, this temperature is the temperature for given [? with ?] the observer at some location of  $\rho$ . So this is for local observer. If you just talk about the  $\eta$ , then the temperature is just  $h$  divided by  $2\pi$  because  $\theta$  is periodical in  $2\pi$ . And if you ask what is the temperature associated with  $\eta$ , it's just  $h$   $\pi$  divided by  $2\pi$ . So any questions regarding this statement?

So now let me [INAUDIBLE] is going to be abstract, so let me elaborate a little bit, and let me make some remarks. So the first is that the choice of vacuum for QFT in the curved spacetime is not unique. So this is a standard feature. This is a standard feature, so if you want to define a quantum fields theory in the curved space time because, in general, in the curved space time, there's no prefer the choice of time. So in order to quantize here, you have to choose a time.

So depending on your choice of time, then you can quantize your theory in a different way. And then you have a different possibility of what is your vacuum. So in other words, in the curved spacetime, in general, the vacuum is observer dependent. So of course, the black hole is the curved spacetime, so the particular continuation procedure we described corresponds to a particular choice, to a special choice, of the vacuum. In the case of a black hole, this has been called to be the Hartle-Hawking vacuum because the Hartle-Hawking first defined it from this Euclidean procedure.

And for the Rindler spacetime, time so we see a little bit later in today's class that this is actually-- the choice of vacuum actually just goes one into the standard of the Minkowski way vacuum. reduced to the Rindler patch. So of course, you can actually choose the vacuum in some other way. For example, say if for a black hole, instead of periodically identify  $\tau$  with this period, we can choose not to identify  $\tau$  at all. Suppose for a black hole, in Euclidean signature we take  $\tau$  to be uncompact. So we don't identify.

So this, of course, runs into a different Euclidean manifold. Then that corresponds to a different Euclidean manifold. And, again, you can really continue the Euclidean theory on this manifold back to [INAUDIBLE] to define our vacuum. And that gives you a different vacuum from the one in which you identified  $\tau$ .

So this gives you a so-called Schwarzschild vacuum. Or sometimes people call it Boulware vacuum because Boulware had worked on it in the early days before other people. And, in fact, this Schwarzschild vacuum is the most natural vacuum. In fact, this is the vacuum that one would get by doing canonical connotation in the black hole in terms of this Schwarzschild time  $t$ .

So let me just emphasize, so this is a curved spacetime. You can just put a quantum fields theory in this spacetime, and then there's a time here which, actually, the spacetime is time dependent. So you have a time-independent Hamiltonian [ $H$  so?] respect to this time. And so, in principal, you can just do a straightforward canonical connotation. And then that connotation will give you a vacuum.

And that vacuum would corresponding to the same as you do analytic continuation to the Euclidean. And don't compact  $\tau$  because there's nothing special. Just like when you do a 0 temperature fields theory in the standard Minkowski spacetime. But when you compactify  $\tau$ , then you get a different vacuum. It is what would be now called a Hartle-Hawking vacuum, and it's what you get a finite temperature.

Is this clear? Yes?

**AUDIENCE:** So in what sense are these different vacuums physical or not?

**PROFESSOR:** Yeah, I will answer this question in a little bit. Right now, I want you to be clear. Right now we have two different vacuum here corresponding to two different ways of doing analytic continuation. And one way you compactify  $\tau$ , and one way you don't compactify  $\tau$ .

And the way you don't compactify Tau is actually the most straightforward way. So if you just do a straightforward canonical connotation using this time, then that will be the vacuum you get. So that's why this is called a Schwarzschild vacuum, OK?

**AUDIENCE:** So do you not have a similarity in writing?

**PROFESSOR:** I will talk about that. So similarly for the Rindler, again, if you do straightforward canonical connotation, in this spacetime with eta has your time, then when you go to Euclidean signature, you want compactify the Euclidean version of eta. Then you won't compactify theta. So if you take the theta to be uncompact, then you get what is so-called Rindler vacuum. So it just, again, can be obtained.

So then you ask them, why do we bother to define those vacuums? Why do we bother to make this identification given that this is the most straightforward thing to do, say, when we consider this quantum fields theory inside curved spacetimes. So here is the key, and so that's where the geometric consideration becomes important.

So, again, let me just first use the black hole as an example case. So in the Schwarzschild vacuum, the corresponding Euclidean manifold is singular add to the horizon. So we explained last time, when we do this [INAUDIBLE], the spacetime is regular only for this periodic identification. For any other possibilities, then you will have a conical singularity add to the horizon.

So this implies the Euclidean manifold singular horizon. Say, if you cut computer correlation function [INAUDIBLE], then you can have single behavior at the horizon. In particular, when you analytically continue to [INAUDIBLE] signature, then that means that physical observables often are singular after the horizon. So we need to continue to [INAUDIBLE] signature.

For example-- which actually I should have maybe the [INAUDIBLE] problem but then forgot earlier. For example, you can check yourself that just take a free scalar fields theory. You compute the stress tensor from this free scalar fields theory, and you find the stress tensor of this theory blows up after the horizon in the Minkowski vacuum.

And if this is not the case, for the vacuum we obtained by doing analytic continuation this way-- so this is not the case for a Hartle-Hawking vacuum. So Hartle-Hawking vacuum just is [? lame ?] for the vacuum we obtained by this analytic continuation procedure for which all observer was regular just, essentially, by definition, by construction.

So essentially by construction because Euclidean manifold is completely smooth, and if you want to compute any correlation function-- for example, you can compute the Euclidean signature and then I need you to continue back to [INAUDIBLE] signature because it's the smooth side of the horizon of the Euclidean signature. I need the continuation back. We also give you a regular function at the horizon there. So similar remarks apply to the Rindler space, and this Rindler vacuum will be singular at this Rindler horizon. It will be singular at Rindler horizon.

So since from the general relativity, the horizon is a smooth place. The curvature there can be very small if you have a big black hole [INAUDIBLE]. So physically, we believe that these Hartle-Hawking vacuums are more physical then, for example, the Schwarzschild vacuum. But Schwarzschild is still often used for certain purposes. But if you want to consider these physical observables, then the Hartle-Hawking vacuum should be the right one to use if you don't want to encounter singularities at the horizon.

Any questions? Yes?

**AUDIENCE:** [INAUDIBLE]?

**PROFESSOR:** No.

**AUDIENCE:** [INAUDIBLE].

**PROFESSOR:** Yeah. Yeah, because, for example, Euclidean Tau is uncompact.

**AUDIENCE:** So the singularity of the physical variables [INAUDIBLE] to become singular to Hartle-Hawking?

**PROFESSOR:** Yeah, that's right.

**AUDIENCE:** [INAUDIBLE]?

**PROFESSOR:** I would say this is the physical reason. If I think about it, I might other not-so-essential reasons. Yeah, but at the moment, I could not think of any others. Yeah?

**AUDIENCE:** So if I understand correctly, when you quantize a theory in curved spacetime, you have to choose a space-like foliation of your spacetime, then you quantize it on that foliation basically. So the problem is that I still don't understand. So the different ground states that you get for different possible quantizations based on different foliations.

Like, to what degree are they compatible versus incompatible with each other? Are they at odds with each other? Like is one more real than another one, or is it just sort of an artifact of your coordinates, and there are things that we can do which are independent of which one we end up with.

**PROFESSOR:** Yeah, so first, in general, they are physically inequivalent to each other. For example, in this case, the Schwarzschild vacuum is physically inequivalent to the Hartle-Hawking vacuum. And actually, with a little bit of work, one can write down the explicit relation between them. And so from the perspective of the Schwarzschild vacuum, the Hartle-Hawking vacuum is a highly excited state and vice versa. So vice versa.

And so, yeah, for local observable, say in the local, it depends. I think the vacuum is more like-- sometimes certain vacuums are more convenient for a certain observer and a certain vacuum is more convenient for that observer. Sometimes it depends on your physical [INAUDIBLE], et cetera. And other questions?

**AUDIENCE:** So we cannot treat these two vacuums as like some transformation?

**PROFESSOR:** Yeah, there is some transformation between them.

**AUDIENCE:** So they stay just the same? Like do [INAUDIBLE]?

**PROFESSOR:** No, they are different states. They are completely different states. You can relate to them in a certain way, but they're different states. Yeah, they're different states.

**AUDIENCE:** If they are different, than why can we only say one is wrong? Because, for example, if it's singular, it's not physical, why don't we just discard it?

**PROFESSOR:** I'm sorry?

**AUDIENCE:** Why we--

**PROFESSOR:** Yeah, physically we do discard it. Physically we don't think the Schwarzschild vacuum is-- so here there's assumption. So the assumption is that we believe that the physics is now single at the horizon. So this is a basic assumption. And if that's the basic assumption, then you should abandon Schwarzschild vacuum as the physical choice of the vacuum. Then you say, then if we really do experiment in the black hole geometry, then what you will discover is the property of the Hartle-Hawking vacuum.

But if our physical assumption is wrong that actually a black hole horizon is singular, then maybe Schwarzschild horizon will turn out to be right. But in this case, we actually don't know. We cannot really do experiments in the black hole, so we cannot really check it.

**AUDIENCE:** But we can measure black hole temperature from infinitely far away, and the Schwarzschild vacuum it would be 0.

**PROFESSOR:** No, but you cannot measure it. We have not been able to measure it.

**AUDIENCE:** And what about the [INAUDIBLE]?

**PROFESSOR:** Yeah, first produce the black hole first.

[LAUGHTER]

Yeah, we will worry about it after we have produced the black hole.

**AUDIENCE:** But if they didn't radiate, they would not decay at all.

**PROFESSOR:** Yeah, then you may also not see it. Anyway, so this can be considered as a physical interpretation of the temperature inside the regularity at the horizon forces us to be in this Hartle-Hawking vacuum. And then the Hartle-Hawking vacuum is like [? a similar ?] state.

OK, so now they may explain why [? they ?] say, they will behave like [? a similar ?] state, OK? So let me go to the second thing I will explain today. It's the physical origin of the temperature.

So, again, I can use either black hole or the Rindler example. The mathematics are almost identical. But I will use the Rindler example because the mathematics are slightly simpler. So I will use the window of spacetime.

So I will explain using the Rindler example. So I will explain at A that this choice of theta to periodically identify theta plus theta plus 2pi [INAUDIBLE] into the choice of a Minkowski vacuum. Then the second thing I will do is I will derive this temperature-- derive the temperature using a different method. So that's I will do today.

So here, when we say, this temperature, we just directly read the temperature from the period of the Tau, OK but I will really derive with the thermal density matrix. And then you will see that this temperature is indeed the temperature with appears in the density matrix. Any questions?

So these two together also amounts to the following statement. So this is an important statement, so let me just write it down, so this A and B also amounts to be the following-- also is equivalent to the following statement. It says, the vacuum inside the Minkowski-- the standard of the Minkowski vacuum where you do your quantum fields theory, the Minkowski vacuum appears to be in a similar state. This temperature which is  $\pi$  divided by  $2\pi$  or  $T$  equal to  $\hbar$  divided by  $2\pi$  in terms of  $\eta$ . OK, it depends on which time you use to a Rindler observer of constant acceleration rate.

So a says, actually, this choice [INAUDIBLE] a periodic of  $2\pi$ , actually we are choosing the Minkowski vacuum. And then the second statement you said, but the stand of the Minkowski vacuum, which appears to be 0 temperature to ordinary Minkowski observer than appears to be a similar state to a Rindler observer. OK, so that's the physical content of this things which we will show. Any questions on this? Yes?

**AUDIENCE:** [INAUDIBLE]?

**PROFESSOR:** It's actually a tricky question. In some sense, they don't really belong to the same Hilbert space. Yeah just when you talk about quantum fields theory, it's a little bit tricky when you have an infinite number [INAUDIBLE]. Yeah, but one can write down the relation between them.

**AUDIENCE:** [INAUDIBLE].

**PROFESSOR:** Yeah, one can write down a relation between them, and then if you take the [ $\eta$  modulus  $\eta$ ] of that vacuum. Then you'll find that it's actually infinite because you have an infinite number that [INAUDIBLE]. It's not possible to normalize that state.

OK, so that's what we are going to show. Hopefully, we will reach it by the end of this hour. But before that, we need a little bit of preparation to remind you of a few things. So once we go through these preparation, then final derivation only takes less than 10 minutes.

So first, he said, they are actually two descriptions of a similar state. So let me remind you. They are two descriptions of a similar state. So we will use harmonic oscillator as an example. So now let's consider a single harmonic oscillator a finite temperature.

So the standard way of doing it is that if you want to compute the expectation value of some operator, of some observable at finite temperature, you just do the standard canonical average. So the  $H$  would be, say, the Hamiltonian of this single harmonic oscillator, and the  $Z$  is the partition function. So can also be written as trace  $x$  and as a thermal density matrix. So the thermal density matrix is  $1/Z$  times  $e^{-\beta H}$ , and the  $Z$  is just the sum of all possible states.

So this is a standard way you would do, say, the finite temperature physics. So this, of course, applies to any quantum systems causal quantum fields theories, et cetera. But actually, this alternative way to do thermal physics-- so this was realized by Umezawa in the 1960s. So he said, instead of considering this thermal density matrix, let me just consider two copies of the same. Let's consider two copies of harmonic oscillator.

Let's double the copy, and then now we have  $H_1$ . Then the foreseeable space for these two copies will be the  $H_1$  of one system tensor product of  $H_2$  and the Hilbert space of the other. And then you will have  $H_1 H_2$ , the Hamiltonian  $H_1 H_2$  associated with each of them. But, of course, these two  $H$ 's are the same.

Then the [INAUDIBLE] system in a typical state would be, say, of the form  $\sum_n m_n$ ,  $a m_n$ ,  $m_1$  [? canceled ?] with  $n$  and 2. So the general state, say, of this doubled system will be like this. So this is two copies of the same system with no interaction with each other.

So now he says, in order to consider thermal physics, let's consider special state defined by the following,  $\frac{1}{\sqrt{z}} \sum_n z^n$ . And he said, now let's consider the following states. So this is an entangled state between the two systems. So this is an entangled state.

So the key observation is that if you want to consider the thermal physics, it says, for any observable, say this  $x$ -- so [INAUDIBLE] 1, which only acts in one of the systems. [INAUDIBLE] act on the first system. So let's consider any such observable.

So now, if you to take the expectation value between the sides, the  $x$  between the sides, so you essentially have a side squared. Then you have  $1/z$ . Then also  $\tau$  will become better  $E_n$ , and then this just becomes  $1/z \sum_n$  [INAUDIBLE] and  $x^n$ . So this is the thermal average. This is sum average.

So if you understand the why, you can also consider because if we are interested only in system 1, then you can just integrate our system 2. We can trace our system 2. Say, suppose we trace our system 2 of this state, then, of course, what do you find? Which is the thermal density matrix of the system 1.

So in other words, the thermal density matrix in one system can be considered as [? entangled ?] pure states of a doubled system. And because we know nothing about the other system, once we trace the 1 to the other system, then you get a density matrix. And this density matrix comes from our insufficient knowledge of the other system. So this is another way to think about this thermal behavior. Do you have any questions on this?

So the temperature arises due to an ignorance of system 2. So because if you now have full knowledge of one of the two, then this would be just the [? size ?] of a pure state. It's just a very special pure state. But if you don't know anything about system 2 and if you consent of only about system 1, then when you trace out system 2 then you get the thermal density matrix for 1.

So let me make some additional remarks on this. I'll make some further remarks. First, this, of course, applies to any-- even though I'm saying I'm using the harmonic oscillator, this, of course, applies to any quantum systems including the QFT's. So the second remark is that this side is actually a very special state. This size is invariant under  $H_1$  minus  $H_2$ ,  $H_1$  minus  $H_2$ . So you can see it very clearly from here.

It says, if you add  $H_1$  and  $H_2$  on this state,  $H_1$  minus  $H_2$  because both  $n$ 's have the same energy, so they just cancel. So this is invariant under this-- yeah, I should say, more precisely, it's [? annihilated ?] by  $H_1$  minus  $H_2$ . And it's invariant under any translation created by  $H_1$  minus  $H_2$ .

**AUDIENCE:** [INAUDIBLE]?

**PROFESSOR:** Yeah, just double the system. It's two copies of the same system. So the third remark now relies on the harmonic oscillator.

For the harmonic oscillator, we can write this  $\psi$  in the following form. So this you can check yourself. I only write down the answer. You can easily convince yourself this is true. Some of you might be able to see it immediately just on the blackboard.

As you actually write this in the-- if you can write this as some explanation of some-- so  $a_1$   $a_2$  are creation operators respectively for two systems. And then the  $0_1$  and  $0_2$  are the vacuum of the two harmonic oscillator systems. And you can easily see yourself, when you expand this exponential, then you essentially just take the power of  $n$ . And then that will give you  $n$ . Yeah, when you act on  $0_0$ , then that will  $n$  a times  $n$ , OK?

So this is normally called a squeezed state. So this tells you that this  $\psi$  is related to the vacuum of the system by some kind of squeezed-- so this is a squeezed state in terms of the vacuum. So this form is useful for the following reason. It's that now, based on these three, one can show it's possible to construct two oscillators, which are  $b_1$  and  $b_2$ , which [INAUDIBLE] the  $\psi$ .

So the  $b_1$  and  $b_2$  are constructed by the following. So, again, this you should check yourself. It's easy to do a little bit of algebra.

So  $\cos \theta$  is equal to  $1 / \sqrt{1 - \beta^2}$ . So the  $\omega$  is the frequency for this harmonic oscillator system and the [INAUDIBLE]. So this  $b_1$   $b_2$  are related to  $a_1$   $a_2$  by some linear transformation.

So what this shows is that [? while ?]  $0_1$  and the  $0_2$  is a vacuum for  $a_1$  and  $a_2$ , and this side is a vacuum for  $b_1$   $b_2$ . So then maybe you can see that this goes [? into ?] a different choice of vacuum. So as we'll see later, the relation between the so-called Hartle-Hawking vacuum and the Schwarzschild vacuum is precisely like this. They just, of course, run into a different choice of alternators. And, in particular, their relation is precise over this form because, say, if you consider some [? three ?] series because the [? three ?] series essentially just reduces to harmonic oscillators.

OK, so this is the first preparation. And those things are very easy check yourself because this is just a single harmonic oscillator. Any questions on this?

Oh, by the way, this kind of transformation is often called Bogoliubov transformation. So the [INAUDIBLE] you assume by this transformation is that in the expression for  $b_1$   $b_2$ , a [? dagger ?] appears here. If there is only a that appears here, then  $b_1$   $b_2$ ,  $a_1$ ,  $a_2$ , they will have the same vacuum because they will [INAUDIBLE] the same states. So the [? nontrivial ?] thing is because of the [? dagger ?] appearing here, so now the state which they are [INAUDIBLE] are completely different and [? graded ?] by this kind of squeezed-state relation. Yes?

**AUDIENCE:** So it's related to the  $H_2$  minus  $H_2$ ?

**PROFESSOR:** I'm sorry?

**AUDIENCE:** The operators like  $b_1$   $b_2$  are somewhat similar to the  $H_1$  minus  $H_2$ ?

**PROFESSOR:** Not really. You can write down  $H_1$   $H_2$  in terms of  $b_1$  and  $b_2$ . You can certainly do that. Yeah, but  $H_1$   $H_2$  is also very simply. You write in terms of  $a_1$  and  $a_2$ .

**AUDIENCE:** If a letter transformation [INAUDIBLE] we get-- never mind.

**PROFESSOR:** Yeah, you can find the  $b_1$  [INAUDIBLE]. Just take the conjugate. Good, so this is the first preparation, which is just something about the harmonic oscillator, which actually can be generalized also to general quantum systems. So the second is I need to remind you a little bit the Schrodinger representation of QFT's.

So normally when we talk about quantum field theory, you always use the Heisenberg picture. We don't talk about wave function. But this equivalent formulation of, of course, quantum fields theory, which you can just talk about wave functional and talk about it in terms of the Schrodinger picture.

So for example, let me just consider a scalar field's theory. Then the Hilbert space, the configuration space, of this theory of this system is just  $\phi(x)$ . You validate it at some given time. Say, you validate it at  $t$  equals to 0, for example. So let me just write it as  $\phi(x)$ .

The configuration space is just all possible  $\phi$  defining the spatial slice. OK, so this your configuration space for the quantum fields theory. And the Hilbert space of the system just given by all possible  $\psi$  functionals of this configuration space variables. So if you feel this is a little bit too abstract to you, then just think of space is discrete, and then you can just write this as some discrete set of variables that them become identical to the ordinary quantum system.

So two things to remind you, two more things. First, just as in quantum mechanics, if you ask the value of, say, a time  $t_2$ , you are in the  $|\phi(t_2)\rangle$  state of this  $\phi$ , this  $\phi(t_2)$  value  $\phi(t_2)$  and the overlap we said  $t_1$  in the  $|\phi(t_1)\rangle$  state of  $\phi$ , this  $\phi(t_1)$  value  $\phi(t_1)$ . This is given by a path integral  $\int \mathcal{D}\phi$ .

You integrate with the following boundary condition. And actually, just  $|\phi(t_1)\rangle$  for this theory. In particular, by taking a linear of this formula, one can write down a path integral representation of the vacuum wave function. So the vacuum wave function, in this case, will be very functional.  $\int \mathcal{D}\phi e^{-iS[\phi]}$ , so if you have a vacuum state, then you just consider the overlap with the vacuum to your configuration space variable.

So I will denote it by  $\phi_0(x)$ . This is the vacuum. And this has a path integral  $\int \mathcal{D}\phi e^{-iS[\phi]}$  is that you compactify your time. OK, so it's where this is your real time, and this is your imaginary time, which I call  $t_e$ . You compactify your time.

And then integrate the path integral or back to imaginary time. So this can be written from the path integral as  $\int \mathcal{D}\phi e^{-S[\phi]}$ . So you go to Euclidean, the but integrate all  $t$ 's more than 0 with the boundary condition  $t_e$  equal to 0x. You can do  $\phi(x)$ , and your Euclidean action.

OK, I hope you're familiar with this. This is where we obtain the vacuum  $|\phi_0\rangle$  function. So if this is not for me to  $|\phi_0\rangle$  say, well, how do you get to the wave function of a harmonic oscillator in the vacuum from path integral? Yes?

**AUDIENCE:**  $|\phi_0\rangle$ ?

**PROFESSOR:** You will see it in a few minutes because I'm going to use those formulas. Yes?

**AUDIENCE:** So one question is that what is the analog of the wave equation for the wave functional? Like what is the  $|\phi_0\rangle$  equation for the wave functional?

**PROFESSOR:** The same thing.  $i \partial_t \phi = H \phi$ .

**AUDIENCE:** OK.

**PROFESSOR:** Yeah, it just has more  $|\phi_0\rangle$  freedom. The quantum mechanics works the same.

**AUDIENCE:** OK.

**PROFESSOR:** So is this familiar to you? If not familiar to you, I urge you to think about the case of a harmonic oscillator. For the harmonic oscillator, that will be the way you obtain the ground state wave function from the path integral is that you first need to continue the system to the Euclidean. And they integrate the path integral all the way from minus Euclidean time equal to infinity to Euclidean time equal to 0.

Let me just write down the-- so the standard of quantum mechanics if you want to right down to the ground state wave function, so that's what you do. Again, you go to the path integral. You go to go to Euclidean, and you really integrate of all  $tE$  smaller than 0. And with the boundary condition that  $x$  [? evaluated ?] at  $tE$  equal to  $tE$  equal to 0 equal to  $x$ . So that gives you the wave function, gives you the wave function.

OK, now with this preparation, that's just some ordinary quantum mechanics. And this is a generalization of that to quantum fields theory just replace that  $x$  by  $\psi$ . Good, so now let's go back to prove the statement we claim you're going to prove, this one, OK?

So now let's come back. So we finished our preparation. Now come back to Rindler space. So it is considered scalar field theory, for example. So let me just remind you, again, this Rindler-- so write down some key formulas here-- so Rindler is the right quadrant of the Minkowski time, which is separated by this right column.

Suppose this is  $X$ . This is  $T$ . Then Minkowski [INAUDIBLE] is minus  $dT$  squared plus  $dX$  squared. And then the Rindler path is minus  $\rho$  squared to  $\eta$  squared plus the  $\rho$  squared.

And so this is the  $\rho$  equal to constant. And this is the  $\eta$  equal to constant. And here is  $\eta$  equal to minus infinity. So here is  $\eta$  equal to plus infinity. So spacetime foliates like this, OK?

And when we go to Euclidean signature for the Minkowski, of course, it's just  $T$  goes to minus  $iT$ . So I call this Euclidean time. For the Rindler, I will do minus  $i\theta$ . This is my notation before.

So in the Euclidean signature, the standard Minkowski just becomes  $TE$  squared plus the  $dX$  squared. And then this Rindler is just  $\rho$  squared [? to ?]  $\theta$  squared plus the  $\rho$  squared. In particular, this  $\theta$  identified to be plus period  $2\pi$ . The Euclidean, under the continuation of Minkowski and Rindler, are actually identical. Both of them are the full two-dimensional Euclidean space.

So if you do the standard of the Minkowski, just replace  $X$  by  $TE$ . But for the Rindler, you just replace it by  $\rho\theta$ . So they just go one into a different foliation, and this goes one into the [INAUDIBLE]. And this one is bound for the Cartesian coordinate.

So the remarkable thing is that even in the Lorentzian signature, the Rindler is only part of the Minkowski spacetime. But once you go to Euclidean, if you do this identification  $\theta \cos \theta + 2\pi$ , they have exactly the same Euclidean manifold, just identical. So this immediately means one thing. It's that all the Euclidean observables just with the trivial coordinate transformation from the Cartesian to the polar coordinates, all the Euclidean observables are identical in two theories.

**AUDIENCE:** What is that down there, "are identical?" What's  $R$  sub  $E$  squared?

**PROFESSOR:** Yeah, this just means the Euclidean two-dimensional spacetime.

**AUDIENCE:** Oh, I see.

**PROFESSOR:** Yeah, let me just right it. This is  $R$  subscript  $E$  squared 2. So this is just a two-dimensional Euclidean space.

So from here, we can immediately lead to the conclusion we said earlier because for Minkowski-- say, if you compute the Euclidean correlation functions, and then [INAUDIBLE] to the Lorentzian signature, what you get is that you get the correlation function in the standard Minkowski vacuum. So for Minkowski and then back to Lorentzian, so the typical observables-- so Euclidean, say, correlation functions just goes to correlation functions in the standard Minkowski vacuum, OK? Yeah, this is just trivial QFT in your high school.

[LAUGHTER]

But with this top statement, we reach a very [INAUDIBLE] conclusion is that for Rindler, when you go back to Lorentzian signature, then that tells you for the Rindler when you do that [INAUDIBLE] continuation and back to the Lorentzian signature, what do you get?

It's that you get a correlation function in the standard Minkowski vacuum. But for observables, restricted to Rindler because it's the same thing. It's just the same function when we do a [? continuation, ?] just you do it differently. It's just on the continuation procedure, it's a little bit different.

And a way to go into Rindler, you just get observable restriction to the Rindler path. So the Euclidean things are exactly the same. So that tells you that the corresponding Lorentzian Rindler correlation function is the same as the correlation function in the standard Minkowski vacuum, but you just restrict to the Rindler patch.

OK, now is our final step. So now let's talk a bit more about the structure of the Hilbert space. So now I have to derive the temperature, but somehow in the Minkowski, there's no temperature.

Here there's no temperature. It's  $T$  equal to 0 from your high school. And here, we must see a temperature. So where does this temperature come from?

So now let's look a little bit at the structure of the Hilbert space. So using that picture-- so let me write here-- so the Hilbert space of the Rindler-- so this is all not in the Lorentzian picture. So the Hilbert space in the Rindler is essentially all possible square integrable wave functional of the  $\phi$ , which defines in the right patch. So when we define this wave functional, we verify at the single time slice.

So that's evaluated at a slice which is  $\eta$  equal to 0, which is just here. So with  $\phi_R$  is essentially just  $\phi$  for  $x$  greater than 0 and the  $T$  equal to 0. This is the right half of the real axis.

So this is a Hilbert space of the Rindler. And, of course, we can also write down the Hamiltonian for the Rindler respect to  $\eta$ . So this is called the Rindler Hamiltonian. And as I said before, you can quantize this here unit, this Hamiltonian, then you will get-- say, you can quantize this Hamiltonian to construct all the excited states, so to me just label them by  $n$ .

So I say, this complete set of eigenstate for  $H_R$ , so the second value will say,  $E_n$ . OK, you can just quantize it, then you can find your full state. In particular, the ground state, when you do that, is what we call the Rindler vacuum. We said before that if you just to straightforward canonical connotation, you get the Rindler vacuum, which is different from the vacuum which we identify  $\theta$  by  $2\pi$ . And this we just straightforward quantize respect to  $\eta$ .

So now let's go back. So this is a structure of the Hilbert space in some sense for this Rindler. So now let's look at the Hilbert space for the Minkowski. So Minkowski is defined-- by Minkowski, I mean, the theory defined for the whole Minkowski spacetime is this  $T$ . Quantize with respect to this  $T$ .

So this would be just standard  $\psi(\phi, x)$ . Now this  $x$  can be anything, and then we also have a Minkowski Hamiltonian defined with respect to  $T$ , to capital  $T$ . OK, so this is the standard of the Minkowski Hamiltonian. And then the vacuum, which I denote to be the Minkowski vacuum is  $M$ . And then vacuum functional  $\psi$  is just  $\psi(x) = \psi_0(M)$ .

So now the key observation here is at  $\psi(x)$ , which this  $\psi(x)$  is for the full real axis, for the full horizontal axis. So this is in some [? contained ?]  $\psi_L(x)$  and the  $\psi_R(x)$ . This goes one into the variable of  $\phi$ . You [? variate ?] it to the right patch, and the value you've added to the left patch.

So the space of  $\psi(x)$ , it's the combination of space of  $\psi_L(x)$  for the left part and space for the  $\psi_R(x)$  part. In particular, this tells us the Minkowski Hilbert space, which is defined as a functional of this full  $\psi(x)$ , should be the tensor product of the Rindler to the right tensor product of the Rindler to the left. Notice that we talk about this right Rindler, but there's a similar Rindler to the left, but the structure of the Minkowski space or Hilbert space, it's the tensor product of the two. In particular, this ground state wave functional, you can write it as  $\psi_L(x)$  and  $\psi_R(x)$ . So this ground state wave functional should also be understood as the functional for  $\phi_L$ .

OK, so now here is my last key. And here is last key. So remember, the ground state wave function will be obtained by doing path integral on the Euclidean half plane. So  $T$  is smaller than 0, OK?

So let me just draw the Euclidean space again,  $X, TE$ . So the wave function can be written as define [?  $tE$  ?]  $x$  [INAUDIBLE] minus  $SE$  Euclidean action. And the lower half plane of the Euclidean space with the boundary condition-- OK, remember from path integral, that's how you obtain the-- so that means you integrate over all this region, path integral over all of this region. Then with boundary condition fixed at here, we do this path integral, you get the Minkowski vacuum wave function.

Now here is the key. So this half space, when I write in this form, you treat this  $T$  as the time, but now let's consider a different foliation. I don't have colored chalk here. I can consider a different foliation. It's that foliation in terms of the  $\theta$ .

Then you just have this foliation, so for each value of  $\theta$ , you have a  $\rho$ . So integration for minus  $\theta$  to  $\theta$ . So if you think about from this foliation-- so think about from this foliation, then this path integral can be written as a following,  $D\phi(\theta, \rho)$ , which you fix  $\phi(\theta) = \phi(\theta = -\pi, \rho)$ .  $\rho$  here when  $\theta$  equal to minus  $\pi$  becomes  $x$ , you could do  $\psi_L(x)$ . And the  $\phi(\theta = \pi, \rho)$ --  $\theta$  equal to 0  $\rho$  equal to  $\psi_R(x)$  and exponential minus  $S$ , so these two should be the same. I just chose the different foliation to do my path integral.

But now if you think from this point of view, this is precisely like that because  $\theta = 0$  and  $\theta = -\pi$  are just two different times in terms of this Euclidean Rindler time. So this you can write it as  $\psi$ , so let me write one more step. So this can be written as  $\psi_2$  [INAUDIBLE] minus  $iH(t_2 - t_1)$   $\psi_1$ .

So this last step can be now written as I can think in terms of the time within the Rindler of time, this can be written as  $\phi_R$  exponential minus  $i$  minus  $i$   $\pi$  because we have a Euclidean time  $H$  Rindler Hamiltonian  $\phi_L$ , OK? So this tells me that Minkowski wave functional can be written as the following-- can be written as  $\phi_R$  exponential  $\pi$   $H_R$   $\phi_L$ .

So now let me expand this in the complete set of states of  $H_R$ . Then this is sum of  $n$ , so if you expand in terms of a complete [INAUDIBLE] to  $H_R$ , then this is, [ $\chi_n$  say,  $\chi_n$ ] wave function. So with this  $\chi_n$  equal to  $\chi_n$   $\phi$  equal to  $\phi$ . So remember, this  $n$  is defined from this Rindler Hamiltonian. Now let me erase this.

So I can slightly rewrite this because I don't like this complex conjugate. I can slightly rewrite this. It's  $\chi_n$   $\phi$  is equal to sum over  $n$  [ $\chi_n$  and then  $\chi_n$ ] minus  $\pi$   $E_n$   $\chi_n$   $\phi_R$  and the  $\chi_n$  and  $\tilde{\phi}_L$ . So  $\chi_n$  and  $\tilde{\phi}_L$  define to be  $\phi$  and  $\star \phi_L$ .

So this  $\chi_n$  and  $\tilde{\phi}_L$  can be considered-- just like in ordinary quantum mechanics, this can be considered to belong to a slightly different Rindler Hilbert space whose time duration is opposite. So when you switch the direction of time, you put a complex conjugate. So this complex conjugate can also be thought of just as a wave function in the Hilbert space with the opposite time direction, with opposite to the Rindler we started with, to the right patch Rindler.

So now this form is exactly the form we have seen before. In this [INAUDIBLE] way to think about thermal state. So this is exactly that because this tells you that the Cartesian in the Minkowski vacuum is the sum of expression minus  $\pi$   $E_n$  with  $n$  Rindler of 1 patch [ $\chi_n$  tensor  $\chi_n$ ] and Rindler of the other one. So this or the other Rindler should be considered Rindler of the left except we should quantize the theory with the opposite time direction, which gives you this Rindler  $\tilde{\phi}_L$  Rindler left.

So this is just precisely the structure we saw before. In particular, if you are ignorant about the left Rindler, you can just trace it out, trace out the left part of this ground state wave function. Then, of course, you just get  $\rho$  Rindler, the thermal density matrix in the Rindler. And this is equal to, of course,  $1$  over Rindler exponential minus  $2\pi$   $H_R$ .

And we see that the  $\beta$  is equal to  $2\pi$ , but this is the  $\beta$  associated with  $\eta$ . As we said, when you change to a different observer, a different  $\rho$ , then you have to go to the local proper time. But this is precisely what we found before.

So now we understand that the thermal nature of this Rindler observer is just precisely that because this observer cannot access the physics to the left patch. But the vacuum state of the Minkowski spacetime is an entangled state between the left and the right. And when you trace out the left part, then you get a similar state on the right part, so that's how the temperature rises. And the same thing can be said about black hole, which we'll say a little bit more next lecture. And let's stop here.